Optimal Design of Experiments for Parametric Identification of Civil Engineering Structures

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Publication date:
1991

Document Version
Publisher's PDF, also known as Version of record

INSTITUTTET FOR BYGNINGSTEKNIK
DEPT. OF BUILDING TECHNOLOGY AND STRUCTURAL ENGINEERING
AALBORG UNIVERSITETSCENTER • AUC • AALBORG • DANMARK

P. H. KIRKEGAARD PH.D. THESIS

OPTIMAL DESIGN OF EXPERIMENTS FOR PARAMETRIC IDENTIFICATION OF CIVIL ENGINEERING STRUCTURES

NOVEMBER 1991 ISSN 0902-7513 R9141

OPTIMAL DESIGN OF EXPERIMENTS

FOR

PARAMETRIC IDENTIFICATION OF CIVIL ENGINEERING STRUCTURES

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Abstract

Optimal design of experiments for parametric identification of civil engineering structures is investigated. Design of experiments for parametric identification of dynamic systems is usually done by minimizing a scalar measure, e.g. the determinant, the trace etc., of an estimated parameter covariance matrix, based on prior knowledge. The experimental conditions available for adjustment, considered in this thesis, are the input signal, sampling rate, the location of sensors and number of sensors. It is found that design of experiments for parametric identification of dynamic systems has been a subject of research during the last decades mainly developed in relation to identification of electrical systems. Design of experiments in relation to civil engineering problems seems to be a subject which only has received little attention during the last decade and will be subjected to research in the future. Further to, an investigation of the applicability of the known methods for design of experiments, a new method is proposed. The investigations of the known methods have shown that they are mainly based on an assumption of statistically independent measurements. However, this assumption can hardly be used in relation to measurements obtained from real engineering structures. This problem is outlined in relation to the optimal location of sensors, and a simple approach taking into account that the measurements in general cannot be modelled as statistical independent random variables is proposed. The investigations have also shown that an experimental design method taking the time, energy and financial resources into account does not exist. The consequence that these kinds of resources are not reflected in the design is that the acquisition of additional information by performing full-scale measurements of a structure can result in unnecessary use of resources. In this thesis a method is proposed making it possible to design optimal experiments where the time, energy and financial resources are reflected. The method is based on a preposterior analysis which is a tool from classical decision theory. One of the contribution of the method is that the optimal number of sensors can be determined.

Preface

The present thesis *Optimal Design of Experiments for Parametric Identification of Civil Engineering Structures* has been made as a part of my Ph.D. study programme during the period from October 1988 to October 1991 at the Department of Building Technology and Structural Engineering, Aalborg University, Denmark. The thesis is made while I have been working at the project "Integrated Experimental and Analytical Investigations of the Dynamic Behaviour of Offshore Structures" which has been performed at Aalborg University since 1988 under the management of Associate Professor, Rune Brincker Ph.D.

I would like to thank my supervisor Associate Professor, Rune Brincker, Ph.D. for his support, guidance and patience during the years. I also thank my other supervisor Associate Professor, John Dalsgaard Sørensen, Ph.D. Department of Building Technology and Structural Engineering, Aalborg University for his guidance and for the loan of his program PRADSS used for computations in this thesis.

The proof-reading has been performed by senior secretary Mrs. Kirsten Aakjær and help by carrying out the figures has be obtained from draughtsman Mrs. Ingrid Christensen. Their careful work is greatly appreciated.

I would also like to thank Ph.D. student Anders Rytter and Ph.D. Jakob Laigaard Jensen for their cooperation at the project "Integrated Experimental and Analytical Investigations of the Dynamic Behaviour of Offshore Structures".

Finally, support from the Danish Technical Research Council is gratefully acknowledged.

Aalborg, October 1991 Poul Henning Kirkegaard.

Resumé (in Danish)

Det er interessant at måle bærende konstruktioners dynamiske egenskaber, da man derigennem kan få belyst de usikre størrelser, samt opnå et erfaringsgrundlag, som kan danne basis for en dokumentering eller revidering af beregningsmodeller. Herved bliver det bl.a muligt på baggrund af fuldskalamålinger, at lave opdaterede sikkerhedsanalyser samt et mere optimalt design for den givne konstruktionstype. I de senere år har der også været en stigende interesse for kontinuerlige fuldskalamålinger, som et alternativ til andre former for konstruktionsovervågning med henblik på f.eks at identificere udmattelsesskader i dynamiske konstruktioner.

Et stort problem med fuldskalamålinger er, at man ikke råder over et tilstrækkeligt pålideligt planlægningsværktøj. Systemidentifikationseksperimenter designes idag i princippet udelukkende intuitivt. Dette betyder en stor risiko for, enten at mangle vigtig information eller at have brugt unødigt mange ressourcer på at sikre sig den nye information, når måleprogrammet er gennemført . Endvidere har man ikke i dag metoder, som kan anvendes til at vurdere et eksperiments informationsindhold i forhold til andres, samt i forhold til eksperimentets samlede omkostning.

På grundlag af disse problemstillinger er formålet med denne afhandling:

- at redegøre for, hvorledes systemidentifikationseksperimenter kan designes optimalt med henblik på at opnå optimal information om estimerede system parametre ved systemidentifikation af bærende konstruktioner
- at udvikle en metode til at vurdere værdien af den forventede nye information, man kan opnå, hvis man udfører et givet måleprogram.

Afhandlingen viser, at der har været arbejdet med design af optimale systemidentifikationseksperimenter de sidste 20-30 år med henblik på at identificere elektriske systemers parametre optimalt. Derimod er det begrænset, hvad der er lavet i relation til bærende konstruktioner. Ved design af et systemidentifikationseksperiment forstås valg af f.eks belastningssignal, samplingsinterval, målepunkters placering etc. I afhandlingen tages der udgangspunkt i de principper og metoder, der er udarbejdet med henblik på at designe optimale systemidentifikationseksperimenter for at identificere elektriske systemer.

Et systemidentifikationseksperiment designes traditionelt, ved at beregne et estimat af en parameter kovarians-matrice for de parametre, man vil bestemme ved eksperimentet. Matricen beregnes på baggrund af ens a priori viden om modellen og værdier af systemparametrene etc. Ved at minimere et skalært mål (determinant, spor) af kovariansmatricen gennem ændring af designvariablerne f.eks belastningen, måleperiodens længde etc. kan et optimal systemidentifikations eksperiment designes.

I kapitel 2 redegøres der for de mest fundamentale principper for systemidentifika-

tion. Der omtales forskellige modelkategorier. Parameter estimering defineres som den del af systemindentifikationsprocessen, der består i at tilpasse en model med nogle givne parametre til et sæt af data. Hovedindholdet i kapitlet er en redegørelse for, hvorledes kovarians-matricen kan estimeres for disse parameterestimater. Ved at antage at en asymptotisk effektiv estimator anvendes, kan kovarians-matricen beregnes på en elegant måde ved den inverse af den såkaldte Fisher informationsmatrice, som forudsætter en sandsynlighedsmodel for målingerne. Kovariansmatricen kan for eksempel også beregnes ved et asymptotisk udtryk, som i et eksempel er vist at være brugbar også for et endeligt antal målinger.

Kapitel 3 redegør for systemidentifikationsteknikker, som normalt anvendes, når en bærende konstruktions systemparametre skal estimeres. Først angives normalt anvendte beregningsmodeller og belastningsmetoder, der anvendes, når en bærende konstruktion skal identificeres. Naturlig belastning ses at være mest anvendt. Dernæst præsenteres kendte teknikker til bestemmelse af systemparametrene i en dynamisk konstruktion. Der omtales såvel simple metoder, hvor systemparametrene bestemmes ud fra f.eks spektralmomenter, som mere komplicerede, hvor systemparametrene bestemmes ud fra f.eks estimater af autocorrelationsfunktioner.

I kapitlerne 4, 5 og 6 diskuteres, hvorledes et optimalt systemidentifikationseksperiment kan designes, når designvariablen er henholdvis belastningssignalet, samplingsinterval/måleperiode eller placering af målepunkter. Kapitel 4 og 5 har til formål at give et overblik over de metoder, der findes til at designe eksperimenter, når enten belastningssignalet eller samplingsintervallet er eneste designvariabel. I kapitel 6 gives derimod en gennemgang af mulige foreslåede metoder til placering af målepunkter. Denne opsummering af metoder til at placere målepunkter ser ud til at være den første, der er lavet.

I kapitel 4 redegøres der først for sammenhængen mellem design af et belastningssignal for styring af en konstruktion og design af et belastningssignal for optimal systemidentifikation, da disse to problemer er nært beslægtede. Sammenhængen består bl.a i, at de to problemer kan løses med løsningsteknikker, som bygger på de samme principper. Endvidere forklares forskellen ved at designe et belastningssignal i tidsområdet og frekvensområdet. Et eksempel med et design i frekvensområdet gives. Til sidst i kapitlet omtales valget imellem at bruge den naturlige belastning eller et optimal designet belastningssignal. Sådant et valg kan gøres, hvis man har en metode til at vurdere værdien af informationsindholdet i forskellige målinger, svarende til forskellige belastningssignaler.

Kapitel 5 omhandler optimal bestemmelse af samplingsinterval og måleperiode. Der præsenteres en metode til løsning af det koblede designproblem med designvariablerne: belastningssignal, samplinginterval og anti-aliasingfilter. Endvidere vises, at et optimal samplingsinterval eksisterer i visse situationer.

Kapitel 6 indeholder en gennemgang af mulige metoder, som er foreslået til optimal placering af målepunkter. Der findes simple metoder, som bygger på intuition. Endvidere findes der metoder, som er baseret på et skalært mål af en parameterkovarians-matricen. En informationsteoretisk metode baseret på et minimum af entropi, vises at være uanvendelig til design af systemidentification-

seksperimenter. I et eksempel afprøves anvendeligheden af forskellige metoder til placering af målepunkter på en simple model af en bærende konstruktion belastet med stokastisk belastning. I kapitlet redegøres der for betydningen af at tage rumlig korrelationen i regning, når målepunkter skal placeres optimalt på en bærende konstruktion, belastet med en stokastisk belastning. Der foreslås en metode til løsning af dette problem.

I kapitel 7 og kapitel 8 forslås en ny metode til design af optimale systemidentificationseksperimenter. Ideen i metoden bygger på, at man bestemmer den forventede opdaterede sandsynlighed for svigt af konstruktionen, hvis eksperimentet udføres. Metoden tager også omkostninger ved eksperimentet i regning.

I kapitel 7 redegøres der for, hvorledes sandsynligheden for konstruktionssvigt kan beregnes, samt hvorledes ny information kan tages i regning ved beregning af en opdateret sandsynlighed for svigt. Sandsynligheden for svigt beregnes v.h.a moderne pålidelighedsmetoder.

I kapitel 8 formuleres selve metoden til design af et optimalt systemidentificationseksperiment. Metoden, der er formuleret på baggrund af beslutningsteori, gør det muligt at vurdere et valgt eksperiments forventede informationsindhold i forhold til eksperimentets samlede omkostninger. Med metoden får man på en rationel måde mulighed for at lave beslutninger vedrørende design af et eksperiment baseret på ens a priori viden. Dermed har man mulighed for f.eks at vælge imellem, om man skal bruge naturlig belastning eller om man skal bruge et optimalt designet belastningssignal. Eller man kan, udover at bestemme den optimale placering af sensorer, også bestemme deres antal, hvilket er nyt.

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Chapter 1

Introduction

Interest in experimental determination of the dynamic characteristics of complex civil engineering structures, such as offshore platforms, bridges, radio towers etc. has increased in recent years. The objectives of these experimental investigations have been to improve and validate computer models or to guide modelling itself, to eliminate the need for extensive analytical modelling, to determine structural parameters that can only be obtained experimentally (e.g. damping) and to attempt to assess structural damage through changes in structural parameters. Certainly, for these reasons and, perhaps, others that one may think of, there is a need to perform experimental investigations of the dynamic characteristics of civil engineering structures.

1.1 Design of System Identification Experiments

Experimental determination of the dynamic characteristics of dynamic systems, named system identification, has grown during the past decades. For many years, the system identification techniques have attracted limited interest in connection with structural engineering applications. One of the factors contributing to this situation is that system identification of dynamic systems has been developed primarily by control engineers and applied mathematicians. Only in the past decade structural engineers have considered this development and recognised its significance. The literature on the system identification problem is extensive. However, the following textbooks Goodwin et al. [1], Ljung [2], Norton [3], and Söderström et al. [4] are fundamental books on system identification.

System identification is the field of modelling dynamic systems from experimental data. More specifically, if one is given measured noisy input/output for a dynamic system, whose description is unknown, the goal of the system identification is to determine a mathematical model, and/or its parameters that describe the system. In some system identification problems, the inputs may be missing. Dynamic

mathematical models are either analytical expressions or graphs that describe the transient behaviour of a system as well as the steady-state effects unlike a static model that only describes the steady state. In other words, a dynamic model has a memory. In general, it is found that system identification is limited to parameter estimation for mathematical models chosen a priori.

It may be noticed that system identification is an experimental approach for constructing mathematical models of dynamic systems while the topic mathematical modelling is an analytical approach. This analytical approach is based on laws from physics, economics etc. Mathematical models can also be constructed as a combination of mathematical modelling and system identification.

Mathematical models are useful because they can provide a description of a physical phenomenon or a process. Scientists often attempt to extract simple and general mathematical descriptions of natural phenomena from complicated information. Further, the models that are found with system identification are usually more accurate, because they are based on measured data rather than assumptions. A dynamic model based solely on mathematical modelling (i.e. physical insight) has been constructed on assumptions.

In figure 1.1, the main steps in a system identification process are shown.

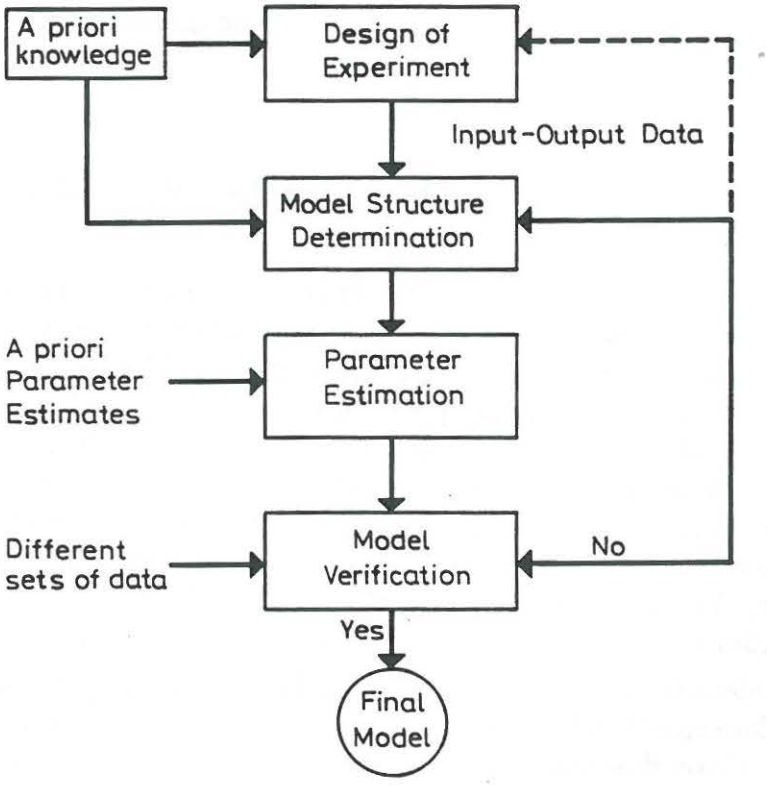

Figure 1.1: System identification process

It is seen from figure 1.1 that the system identification process has a natural logical flow. In general, the steps in figure 1.1 could be developed into following states:

- The model purpose, inputs and outputs, range, scope and accuracy are specified. This provides a target for the system identification project.
- A number of pilot tests is performed to obtain information that is required for the design of the full tests.
- The system identification experiment is designed.
- The test signals are supplied to the system and the data that will be used for the actual model estimation are recorded.
- The recorded data are prepared for the model estimation algorithm.
- System identification algorithm are applied to compute model estimates and estimates of the parameters in the model if it is a parametric model.
- The estimated models are evaluated.
- Each of the above stages is repeated until the targets have been met. But the further back one goes the more expensive it gets.

It is seen that the result of a system identification experiment will be influenced by (at least) the following four factors:

- *The system* $\mathcal{S}$ that in loose terms is an object in which variables of different kinds interact and produce observable signals.
- *The model structure* $\mathcal{M}$.
- *The identification method* $\mathcal{I}$ where a large variety has been proposed in the literature.
- *The experimental conditions* $\mathcal{H}$ describe how the system identification experiment is carried out.

It should be noticed that of the four concepts $\mathcal{S}$, $\mathcal{M}$, $\mathcal{I}$ and $\mathcal{H}$, the system $\mathcal{S}$ must be regarded as fixed. The experimental conditions $\mathcal{H}$ are determined when the data are collected from the process. $\mathcal{H}$ can often be influenced to some degree by the user. However, there may be restrictions that prevent a free choice of the experimental conditions $\mathcal{H}$. Once the data are collected, the user can still choose the identification method $\mathcal{I}$ and the model structure $\mathcal{M}$. Several choices of $\mathcal{I}$ and $\mathcal{M}$ can be tried on the same set of data until a satisfactory result is obtained, i.e. arrived at a particular model: the one in the set that best describes the data. Then testing whether this model is good enough remains.

1.1.1 Design Variables

Design of a system identification experiment concerns the selection of the experimental conditions for the experiment so that it contains a maximum of information of the properties of the system that are pertinent to a particular application. Practically, the experimenter has to be decided upon the purpose, and among other things which variables to measure and how to measure them. These decisions will be taken under various physical, technological and economic constraints and be conditioned upon the amount of prior information available.

In the case of design of an experiment for system identification of a dynamic system, the experimental conditions available for adjustment include:

- input excitation signal (amplitudes, spectra)
- sampling rate
- filtering of the data before sampling
- measuring time period
- spatial location and number of sensors and actuators

The experimental design must take account of the constraints on the allowable experimental conditions, e.g. amplitude constraints on inputs/outputs, total time available for the experiment, maximum sampling rate etc. Further, typical constraints that may be met in practice are the availability of sensors/actuators, filters and hardware/software for signal generation, data storage and analysis. These constraints on the allowable experimental conditions are stipulated during the planning of the system identification experiment. This planning must take account of several factors, including:

- the object of the experiment and the intended application of the results
- the class of models to be used
- the identification methods
- the extent of the prior knowledge about the system.

The choice of the experimental conditions may appear to be fairly straightforward, however, making the wrong choices will have major effect on the quality of the final identified models/parameters. Therefore, it is worthwhile to design the experiment carefully so as to obtain data that are sufficiently informative since repeated data acquisition can be a costly and time-consuming procedure.

The problem of experiment design has been given much attention in the literature. The theory of the design of static experiments originated in the early thirties, see e.g. Fisher [5], and has been considerably developed in the statistical literature after the second world war. Fedorov [6] and Silvey [7] can be mentioned as basic references. To a certain extent the topic was regarded as being largely of theoretical

interest and as having little value for the practising statistician.

However, the models considered in the statistical literature are generally static and their applicability to dynamic models has become clear only recently.

The first attempt in engineering literature of experiment design for system identification of dynamic systems seems to be in the early sixties according to Mehra [8]. Most of the literature considering experiment design for dynamic system identification has dealt with design of optimal input signal for obtaining accurate parameter estimates. The survey paper Mehra [8] presents a profound discussion of input signal design for dynamic system identification. The sampling rate is another experimental condition available for adjustment treated in the literature. Just recently literature has appeared considering experimental conditions available for adjustment besides the input signal and sampling rate. E.g. the problem of optimal positioning of sensors and numbers of sensors has received attention.

The problem of experiment design can be regarded as a generalisation of the problem of optimal input signal design that has been comprehensively treated in the literature. Representative surveys of the problem of experiment design for dynamic system identification are given in the system identification textbooks Goodwin et al.[1], Kalaba et al. [9], Ljung [2], Norton [3] and Söderström et al. [4]. Each of these books has a chapter dealing with the problem of experiment design for dynamic system identification. A more comprehensive treatment of this subject is found in the textbook Zarrop [10] that is concerned with the problem of experiment design for the efficient identification of a linear single input, single output system. In this book, the experimental conditions for adjustment considered are input signal and sampling rate. Beyond these textbooks many research papers exist, mainly on the problem of optimal input design for system identification. Especially, the paper Goodwin [11] may be noticed as a contribution to the literature concerned with experiment design for dynamic system identification.

In general, the effects of the experimental conditions available for adjustment are closely interrelated and a joint design is required. However, most of the existing literature consider the experimental conditions separately to give insight into their individual effects upon the information extracted from a given set of data. Few, however, consider the joint design problem, see e.g. Goodwin et al. [1]. It may be noticed that almost all papers concerning design of experiments for parametric identification of dynamic systems are devoted to the case of experiments, in which, the measurements are modelled as stochastic independent random variables.

1.1.2 Design Criteria

For input design, or experiment design purposes, it is normally assumed that the accuracy of the parameter estimates is most conveniently expressed in terms of the parameter covariance matrix $\overline{\overline{C}}_{\hat{\bar{\theta}}_N}$. Since the parameter estimate $\hat{\bar{\theta}}_N$ of the parameter $\bar{\theta}$ to be estimated from the experiment is dependent on random processes the accuracy of $\hat{\bar{\theta}}_N$ must be considered in a statistical sense.

It may be noticed that many authors postulate the existence of an asymptotically efficient unbiased estimator as a basis for the experiment design, implying that there is a lower bound, the Cramer-Rao lower bound, on the achievable covariance of the parameter estimate $\hat{\bar{\theta}}_N$ irrespective of the estimator algorithm used, provided it is unbiased. Further, they assume that a probabilistic model is known that means that the covariance is given by

$$\overline{\overline{C}}_{\hat{\bar{\theta}}_N} \geq \overline{\overline{J}}^{-1}(\mathcal{H}) \tag{1.1}$$

where $\overline{\overline{J}}$ is the Fisher information matrix, see e.g. Goodwin et al. [1], which is a function of the experimental conditions $\mathcal{H}$. The inverse of the Fisher information matrix is the well-known Cramer-Rao lower bound, see e.g. Goodwin et al. [1].

The reason why, the parameter covariance is estimated based on the inverse of the Fisher information matrix is that the exact covariance of a particular estimator can be difficult to obtain. On the other hand, the Fisher information matrix is easy to estimate if the measurements can be modelled as stochastic independent random variables.

Design based on the Fisher information matrix also leads to some simplification of the experiment design problem. The determination of an optimal experiment leads to a highly complex optimization problem, requiring the simultaneous choice of identification algorithm, system parameterisation, sensor location, input excitation signal, etc. If it is assumed that the choice of identification algorithm is restricted to the class of efficient estimators, the choice of identification algorithm is uncoupled from the overall experiment design. This is a consequence of the fact that the covariance of the parameter estimates for an efficient estimator can be directly calculated in terms of the Cramer-Rao lower bound, or equivalently, the inverse of the Fisher information matrix. This leads to a great simplification, since the minimum variance given by the Cramer-Rao lower bound can be easily computed in a number of estimation problems.

For the purpose of comparing different informative experiments it is necessary to have a measure of the applicability of the experiment. A logical approach is to choose a measure related to the expected accuracy of the parameter estimates to be obtained from the data collected. Clearly, the parameter accuracy is a function of both experimental conditions $\mathcal{H}$ and the parameter estimator. Formally, the problem of optimal experiment for parameter estimation could be stated as

$$\min_{\mathcal{H}} \mathcal{A}\left(\overline{\overline{C}}_{\hat{\bar{\theta}}_N}(\mathcal{H})\right) \tag{1.2}$$

where $\mathcal{A}(\cdot)$ is a scalar function of the covariance matrix. $\mathcal{H}$ is a set of all possible informative experiments. Typically, such scalar functions are, see e.g Goodwin et al. [1], Zarrop [10], Mehra [8] and Ljung [2],

$$\mathcal{A}\left(\overline{\overline{C}}_{\hat{\theta}_N}(\mathcal{H})\right) = \det[\overline{\overline{J}}^{-1}(\mathcal{H})] \qquad (D-OPTIMUM) \tag{1.3}$$

$$\mathcal{A}\left(\overline{\overline{C}}_{\hat{\theta}_N}(\mathcal{H})\right) = \text{trace}[\overline{\overline{W}}\,\overline{\overline{J}}^{-1}(\mathcal{H})] \qquad (A-OPTIMUM) \tag{1.4}$$

$$\mathcal{A}\left(\overline{\overline{C}}_{\hat{\theta}_N}(\mathcal{H})\right) = \max \lambda_i(\overline{\overline{J}}^{-1}(\mathcal{H})) \qquad (E-OPTIMUM) \tag{1.5}$$

where $\lambda_i(\overline{\overline{J}}^{-1}(\mathcal{H}))$ denotes eigen values of matrix $\overline{\overline{J}}^{-1}(\mathcal{H})$. $\overline{\overline{W}}$ is a weighting matrix. In (1.3), (1.4) and (1.5) it is assumed that the covariance matrix is determined by the inverse of the Fisher information matrix.

In Zarrop [10] and Pazman [12] a more detailed discussion of design criteria related to experiment design is given.

The choice between the above criteria depends on the intended application. One should be careful of designs that depend critically on the criterion used for the design purpose, since this indicates that the best experiment for one application will be very different from the best experiment for another application, see Goodwin [11]. From a computational point of view (1.3) and (1.5) may be easier to deal with, since they do not require inversion of the information matrix $\overline{\overline{J}}$. However, the choice of a criterion is often not critical, since it is usually the case that a good experiment according to one criterion will be deemed good by other criteria, see Goodwin et al. [1]. Naturally, this will only be true for sensible chosen criteria. In the literature a criterion such as

$$\mathcal{A}\left(\overline{\overline{J}}^{-1}(\mathcal{H})\right) = \text{trace}(\overline{\overline{J}}(\mathcal{H})) \tag{1.6}$$

has been stated. This criterion is to be viewed with suspicion, since it can lead to the design of experiments in which parameters are unidentifiable, see Goodwin [11].

One drawback by using the above design criteria for experiment design is that $\overline{\overline{J}}$ generally depends on the true parameter values $\overline{\theta}_0$ that are unknown prior to the experiment. If $\overline{\theta}_0$ is known there would be no design problem to solve. However, to solve the design problem the true model must be known. In practice the design problem is dealt with by replacing $\overline{\theta}_0$ by the best prior estimate of the parameters, perhaps obtained from a pilot experiment. From a Bayesian point of view, a prior probability density function for $\overline{\theta}_0$ can be assigned and the prior average of $\overline{\overline{J}}$ can be considered, see e.g. Goodwin et al. [1]. However, a compromise must be made between designing the best experiment and making the design sensitive to the parameter values on which the design is based. In practice, useful designs can be

obtained simply by evaluating (1.3), (1.4) or (1.5) at a representative value of $\hat{\overline{\theta}}_N$ and then check the sensitivity of the design to other parameter values.

However, the basic idea underlying the experiment design theory is that design, e.g. an input design, should be chosen to make a scalar measure of the inverse of the Fisher information matrix as small as possible. By doing so a design is chosen to get as much information as possible about the parameter vector $\overline{\theta}$ in a Fisherian sense. This information is retrievable at least when the number of sample points N is large and estimation is made by an asymptotically efficient unbiased estimator.

It may be noticed that input signal design, or experiment design, can also be made so that it is optimal in other senses than Fisherian sense. If a so-called prediction error estimator is used, and stationary processes are considered, then a design based on an asymptotic expression for the parameter covariance can be made. Such, a design is also optimal in Fisherian sense if the prediction errors are Gaussian distributed. This is a consequence of the fact that the covariance given by the asymptotic expression is going against the Cramer-Rao lower bound if the prediction errors are Gaussian distributed, see e.g. Goodwin et al. [1]

The above discussion has given an introduction to the topic "design of experiments" where the design is based on minimizing a scalar measure of the covariance matrix of the parameter estimates given by the inverse of the Fisher information matrix. This approach is used in most of the literature on experiment design for parameter identification of dynamic systems. However, it is important to remember that the calculations are based on prior information.

1.2 Scope of the Work

In the previous section the principle of system identification of dynamic systems has been presented. It is explained that it is a topic mainly developed in electrical engineering and just in the past decade structural engineers have considered this development and recognized its significance. However, the topic "design of experiments" has not received the same interest from structural engineers as the topic "system identification"

The aim of this thesis is:

- to investigate methods available for design of experiments for system identification of linear dynamic civil engineering structures. Attention is focused on parametric identification experiments.

It may be noticed that a number of books and papers dealing with experiment design for dynamic system identification consider e.g. electrical systems, biological systems, mechanical systems or economical systems. This thesis deals with the

problem from a civil engineering point of view. This implies that the investigations of the methods will be concerned with the applicability of the known methods for design of experiments for parametric identification of civil engineering structures.

The thesis will mainly deal with the following experimental conditions available for adjustment:

- input signal
- sampling rate
- measuring time period
- number and location of sensors.

Application of methods for design of experiments for parametric identification is clearly most useful in areas where experimentation is expensive, such as identification of e.g. offshore structures. However, if the cost of the experiment is not reflected into the design, a full-scale measuring of a structure can result in unnecessary use of resources. When additional information is obtained about the uncertain parameters in a model for an existing structure, e.g. by performing a system identification experiment, the reliability of the structure will be changed. If such a change is negligible and an expensive experiment has been used to obtain the additional information it is considered as an unnecessary use of resources.

Therefore, it is also the aim of this thesis:

- to develop a method which can be used to design system identification experiments. The method should reflect two things. Firstly, the cost of the experiment. Secondly, the expected value of information which can be obtained if the experiment is performed.

1.3 Reader's Guide

In this section the organization of the thesis is presented in order to give the reader an overview.

The thesis has three parts:

- chapters 2 and 3 are concerned with the principles of system identification,
- chapters 4, 5 and 6 deal with different methods for design of experiments,
- chapters 7 and 8 are devoted to present a method for design of experiments based on the expected updated structural reliability.

After the above presentation of experiment design, chapter 2 is made to outline the fundamental principles of system identification, such as parameter estimation, choice of model etc. However, the chapter is mainly written to explain how estimates of the parameter covariance matrix can be obtained, e.g. by the Fisher information matrix.

Chapter 3 is devoted to present the most frequently used system identification techniques in civil engineering. Normally used models and excitation types are outlined. After that, a presentation of different frequency and time domain identification techniques is given.

After giving the bases for parametric identification of dynamic systems and estimation of the parameter covariance matrix, the problem of experiment design is considered.

Chapter 4 is concerned with optimal design of input signal in frequency domain and time domain, respectively.

In chapter 5 the problem of optimal determination of sampling interval and number of data is considered.

Chapter 6 is devoted to the optimal sensor location problem. The chapter presents different methods available for determination of the optimal locations. The applicability of these methods is investigated in relation to civil engineering structures.

After chapters 4, 5 and 6 have shown how design of experiments can be made using traditional techniques a new method for design of experiments for parametric identification of civil engineering structures is proposed in chapter 7 and 8.

Since the method is based on the expected updated structural reliability chapter 7 is devoted to present the techniques available to estimate the reliability of structural systems. Further, it is explained how new information from system identification experiments can be included in reliability calculation so that an updated reliability can be estimated. The chapter establishes the connection between experiment design variables and the expected updated reliability. In chapter 8 this connection is applied to an experiment design method. The method is based on decision theory which is outlined.

Finally, in chapter 9, an overall summary and main conclusion of the thesis are given.

1.4 References

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Chapter 2

Outline of System Identification

System identification is a process for determining the relationship between cause and effect in a physical system. More specifically, if one is given the input and the corresponding output for a system, whose description is unknown, the goal of system identification is to find a mathematical model that describes the system. In general, it is found that system identification is limited to parameter estimation in a model chosen a priori.

In this chapter, the basic techniques of system identification for parameter estimation of dynamic systems are outlined based on Goodwin et al. [1], Ljung [2], Norton [3] and Söderström et al. [4] that are fundamental books on the system identification problems. A complete outline will not be given but some basic problems are considered. The main emphasis of the chapter is put on the problem of estimating the parameter covariance matrix, since the basis of experiment design for parameter estimation is a scalar measure of this matrix. In chapter 3 system identification techniques for parameter estimation of civil engineering structures are discussed. First, in section 2.1, different models are presented. Next, in section 2.2, the term parameter estimation is defined and experiment design problems associated with parameter estimation are discussed. In section 2.3 different parameter estimation techniques are considered . The asymptotic properties of the parameter estimates are discussed in section 2.4. Among other things the problem of estimating the parameter covariance matrix will be considered. In section 2.5 an example is given to investigate the applicability of the parameter covariance estimating methods. Section 2.6 presents a Bayesian approach to parameter estimation and section 2.7 is concerned with model structure selection and model validation.

2.1 Model Categories

The first step in system identification is to determine a model structure $\mathcal{M}$, the

target for the project, within which the search for a suitable model has to be made. This is an important and also a difficult choice of the system identification procedure. It is here that a priori knowledge, engineering intuition and insight have to be combined to obtain a satisfactory model.

2.1.1 Types of Dynamic Models

Dynamic models can be divided into three types. Physical models, black-box models and a mixture of the two, grey-box models.

- *Physical models* are constructed from basic physical laws and general information of the system such as dimensions, weight etc. A set of differential or difference equations with unknown physical parameters is derived from the physical theory.
- *Black-box models* are established without any reference to the physical background. The prime idea is to obtain flexible models that can accommodate a variety of systems, without looking into their internal structure. Only the input-output behaviour is modelled.
- *Grey-box models* are a set of models with adjustable parameters with a physical interpretation. These models are constructed as a combination of mathematical modelling and system identification.

2.1.2 Classification of Dynamic Models

Mathematical models of dynamic systems can be classified in various ways:

- *Single input-single output (SISO) models/multivariable models(MIMO).* SISO models refer to processes where a description is given of the influence of one input on one output. When more variables are involved a multivariable model (MIMO) is obtained.
- *Linear models/nonlinear models.* A model is linear if the output depends linearly on the input and possible disturbance; otherwise it is non-linear.
- *Parametric models/non-parametric models.* A parametric model is described by a set of parameters while a non-parametric model may consist of a function or a graph which is not parameterised by a finite-dimensional parameter vector. A numerical transfer function is an example on a non- parametric model.
- *Time-invariant models/time varying models.* Input-output relations do not vary with the time in a time invariant model and are easier to analyse than a time varying model.
- *Time domain models/frequency domain models.* Typical examples of time do-

main models are differential and difference equations, while a spectral density is an example of a frequency domain model.

- *Discrete time models/continuous time models.* A discrete time model describes the relation between inputs and outputs at discrete time points. Normally it is assumed that these points are equidistant and the time between two points will be used as a time unit. Although most physical systems are continuous their behaviour is usually determined from records of outputs sampled at discrete instants.
- *Lumped models/distributed parameter models.* Lumped models are described by or based on a finite number of ordinary differential or difference equations. If the number of equations is infinite or the model is based on partial differential equations, then the model is called a distributed parameter model.
- *Deterministic models/stochastic models.* For a deterministic model the output can be exactly calculated when the input signal is known. In contrast, a stochastic model contains random terms that make an exact calculation impossible. The random terms can be seen as a description of disturbances.

In the literature one special family of dynamic models has been far more developed than the rest, namely linear, lumped, time-invariant dynamic models. Although these represent severe idealisations of the systems they prove very useful in practice. The reason is that they are simple to identify, analyse and understand.

Historically, system identification based on frequency domain models seemed to dominate the theory and practice of system identification up to the sixties. From the end of the sixties and onwards the interest in system identification based on time domain models has increased and now literature on system identification is very much dominated by time domain methods. System identification based on time domain models and frequency models, respectively can be considered as two complementary approaches. System identification based on frequency models often gives good insight into the properties of the data and the system. Such information combined with engineering intuition is valuable for decisions of type and complexity of models to be used for further analysis of the data. Therefore, a frequency domain approach can be a good first step in the data analysis procedure. Often the intended use of the model as well as accuracy requirements on parameter estimates motivate the use of a time domain model and corresponding system identification procedure. In Ljung et al. [5] the basic features of system identification based on time and frequency domain approaches are highlighted. Further, relationships between the two approaches are explained. Time domain approaches versus frequency domain approaches are also discussed in e.g. Prevosto et al. [6] and Davies et al. [7].

2.2 Parameter Identification and Estimation

In section 2.1 it is mentioned that an estimate of e.g. the autospectrum is very useful and might give much insight into the system under consideration. From such a spectrum parameters describing the system can be obtained by direct examination of the peaks. However, the direct approach of parameter estimation may fail in many situations. Other, more systematic ways of fitting the data are then needed to estimate the parameters, i.e. parametric dynamic models may be considered. In the time domain, a stochastic model is fitted to the data in a time-series form while in the frequency domain e.g. the power spectrum of the random process of the stochastic model is fitted to the power spectrum of the data.

Since the measurements are always contaminated with noise and other limitations, the identification of parameters can at best be an estimation, therefore the term **parameter estimation**. The technical meaning of the word noise is unwanted signal or disturbance. From this point of view, any error in system modelling as well as that in input and output measurements may be defined as noise. Since system identification uses experimentally observed or measured data, the error or noise generated by the measuring instrument is obviously of great importance. It may be noticed that two types of experimental error may arise: random error and bias error. The difference between these errors is that the expected or mean value of the random error is zero, whereas that of the system, bias error, is not. As their names imply, random error is usually traceable to environmental noise, while the bias error is caused by bad modelling. For system identification applications, the bias error is generally damaging and difficult to eliminate. Random error, on the other hand, can often be reduced by averaging techniques.

2.2.1 Parameter Estimation

In this section, the problem of parameter estimation in the time domain is considered for a SISO model. The results mentioned will in general also apply to more complicated problems where e.g. MIMO systems are considered.

The characteristic feature of a parametric time domain method is that a model structure $\mathcal{M}$ is selected. $\mathcal{M}$ describes a set of models $\mathcal{M}^*$ within which the best is sought for. The set is supposed to be parameterised by a finite-dimensional vector $\overline{\theta}$. An individual model within the set is denoted $\mathcal{M}(\overline{\theta})$. The search within the model structure $\mathcal{M}$ for the model $\mathcal{M}(\hat{\overline{\theta}}_N)$ that best describes the data becomes a problem of estimating $\hat{\overline{\theta}}_N$. $\hat{\overline{\theta}}_N$ is an estimate of the parameter vector $\overline{\theta}$ based on a batch Z^N of N measured input-output data from the system

$$Z^N = [y(t), u(t)] \qquad t = 1, 2, ..., N \tag{2.1}$$

where $y(t)$ and $u(t)$ are input and output, respectively, from the system at the sampling time t. I.e. it is assumed that the data is collected at discrete time and therefore a discrete time is considered. The data set Z^N will depend on the system

$\mathcal{S}$ that has generated the data and the experimental conditions $\mathcal{H}$ that were at hand when the data were recorded.

Given Z^N and the model structure $\mathcal{M}$ it is possible to select the best model $\mathcal{M}(\hat{\bar{\theta}}_N)$. This is typically achieved by minimizing a criterion function of the form

$$V_N(\bar{\theta}, Z^N) = \frac{1}{N}\sum_{t=1}^{N} l(t, \bar{\theta}, \epsilon(t, \bar{\theta})) \tag{2.2}$$

where $l(\cdot,\cdot,\cdot)$ is a scalar measure of the size of the prediction error $\epsilon(t, \bar{\theta})$

$$\epsilon(t, \bar{\theta}) \triangleq y(t) - \hat{y}(t|\bar{\theta}) \tag{2.3}$$

and $\hat{y}(t|\bar{\theta})$ denotes a prediction of $y(t)$ based on a predictor model and the parameter vector $\bar{\theta}$.

$\epsilon(t, \bar{\theta})$ describes the deviation in the measured data from the estimate of the system $\mathcal{M}(\hat{\bar{\theta}}_N)$.

It should be noted that the criterion (2.2) applies equally to linear and non-linear dynamic models. The only requirement is that the models provide a way of computing the prediction errors $\epsilon(t, \bar{\theta})$. Further, if the system $\mathcal{S}$ considered is oscillating in a non-linear manner and the model structure $\mathcal{M}$ is linear, then the parameter estimate $\hat{\bar{\theta}}_N$ will converge to the best linear approximation to the true dynamics in the sense of minimum distance between the true dynamics of the best system and the linear dynamics of the estimated model. Convergence of $\hat{\bar{\theta}}_N$ will be discussed in section 2.4.

2.2.2 Parameter Estimation Experiment Problems

In this section, the concepts of identifiability and informative experiments will be discussed. A more thorough discussion of these concepts of identifiability can be found in e.g. Ljung [2], Söderström et al. [4], Söderström et al. [8] and Goodwin [9].

The resulting estimate obtained from (2.2), when an identification method $\mathcal{I}$ is applied to a parametric model structure $\mathcal{M}$, can be denoted

$$\hat{\bar{\theta}}_N(N; \mathcal{I}, \mathcal{M}, \mathcal{H}, \mathcal{S}) \tag{2.4}$$

Clearly, the estimate will depend not only on $\mathcal{I}$ and $\mathcal{M}$ but also on the number of data points N, the true system $\mathcal{S}$ and the experimental conditions $\mathcal{H}$.

The system $\mathcal{S}$ is said to be system identifiable under $\mathcal{M}$, $\mathcal{I}$ and $\mathcal{H}$ if

$$\hat{\bar{\theta}}_N(N; \mathcal{I}, \mathcal{M}, \mathcal{H}, \mathcal{S}) \rightarrow D_T(\mathcal{S}, \mathcal{M}) \;\; w.p.1 \;\; as \;\; N \rightarrow \infty \tag{2.5}$$

The set $D_T(\mathcal{S}, \mathcal{M})$ consists of those parameter vectors for which the model structure $\mathcal{M}$ gives a perfect description of the true system $\mathcal{S}$. Three situations can occur

- The set $D_T(\mathcal{S}, \mathcal{M})$ may be empty. Then no perfect description of the system can be obtained in $\mathcal{M}$, no matter how the parameter vector is chosen. One can say that the model structure has too few parameters to describe the system adequately. This is called *underparametrization.* The set $D_T(\mathcal{S}, \mathcal{M})$ is empty in the case $\mathcal{S} \notin \mathcal{M}$.
- The set $D_T(\mathcal{S}, \mathcal{M})$ may consist of one point. This will then be denoted $\bar{\theta}_0$. This is the ideal case; $\bar{\theta}_0$ is called the true parameter vector.
- The set $D_T(\mathcal{S}, \mathcal{M})$ may consist of several points. Then there are several models within the model set that give a perfect description of the system. This situation is sometimes referred to as *overparametrization.*

The system $\mathcal{S}$ is said to be parameter identifiable under $\mathcal{M}$, $\mathcal{I}$ and $\mathcal{H}$ if it is system identifiable and $D_T(\mathcal{S}, \mathcal{M})$ has exactly one point. This is the ideal case mentioned above. If the system is parameter identifiable then the parameter estimate $\hat{\bar{\theta}}_N$ will be unique for large values of N and also consistent,i.e. $\hat{\bar{\theta}}_N$ converges to the true value, as given by the definition of $D_T(\mathcal{S}, \mathcal{M})$.

The convergence of the parameter estimate $\hat{\bar{\theta}}_N$ to the set $D_T(\mathcal{S}, \mathcal{M})$ (system identifiability) is a property that depends on the identification method $\mathcal{I}$. This is a most desirable property and should hold for as general experimental conditions $\mathcal{H}$ as possible. It is then "only" the model parameterisation or model structure $\mathcal{M}$ that determines whether the system is also parameter identifiable. It is of course desirable to choose the model structure so that the set $D_T(\mathcal{S}, \mathcal{M})$ has precisely one point.

It is seen that identifiability is a concept that is central in identification problems and that, roughly speaking, the problem is whether the identification procedure will yield a unique value of the parameter vector $\bar{\theta}$, and/or whether the resulting model is equal to the true system.

The information about the true system is included in the data set Z^N (the experimental conditions) that is to be fit to a model structure $\mathcal{M}$ describing a set of models $\mathcal{M}^*$ within which the best one is sought for. If the data set Z^N, or Z^∞ since $N \to \infty$, allows discrimination between any two models in the set $\mathcal{M}^*$. Ljung [2] has introduced the concept of data sets that are informative enough with respect to the model set $\mathcal{M}^*$. This terminology is transferred to identification experiments by calling an experiment informative enough if it generates a data set that is informative enough. If an identification experiment generates data allowing discrimination between any two different models in a model structure it is said to be an **informative experiment.** For an informative experiment the experimental conditions $\mathcal{H}$ must be chosen so general that no incorrect model will

be input-output equivalent to the true system under $\mathcal{H}$. This essentially means that the input should excite all modes of the system. In the literature such an input is termed persistently exciting see e.g. Söderström et al. [4]. In chapter 4 this problem will be discussed further.

From the above clearly a primary goal is to design experiments for parameter estimation which provide sufficient information by which model ambiguities can be ruled out.

2.3 Prediction Error Methods

The parametric estimation method in section 2.2 is based on the simple principle that a particular model of the best process is a good model if it allows us to obtain good prediction of the future outputs of the process, i.e. the prediction errors $\epsilon(t, \overline{\theta})$ become as small as possible. Normally, this way of estimating $\overline{\theta}$ is called the prediction error identification methods (PEM). This family of methods contains as special cases several well known and commonly used identification methods. Within the prediction error class, this means the choice of the criterion function $l(\cdot, \cdot, \cdot)$.

2.3.1 Least Squares Estimation

One choice of function $l(\cdot, \cdot, \cdot)$ that is most commonly used is the least squares criterion, i.e. the criterion function is given by

$$V_N(\overline{\theta}, Z^N) = \frac{1}{2N} \sum_{t=1}^{N} \epsilon^2(t, \overline{\theta}) \tag{2.6}$$

This criterion function gives a class of least squares identification methods. However, in the engineering literature this name is usually reserved for identification of a particular set of parameterised models $\mathcal{M}(\overline{\theta})$ which are linear in the parameter vector $\overline{\theta}$. Then the least squares estimate $\hat{\overline{\theta}}_N$ can be calculated analytically.

With the exception of some special cases, the minimum of the criterion function (2.6) cannot be found analytically. Therefore, it is necessary to employ numerical search techniques for the problem of (2.2) known as the nonlinear least-squares problem in numerical analysis. Such techniques are usually based on searching in the gradient direction (steepest descent direction) of the function (2.6) or in a direction that is related to the gradient direction.

It is seen that the computational side of system identification will be very similar to that of non-linear optimization. In particular, there is the problem of local minimum of the criterion function (2.2), since in general one has to expect that a non-linear function such as (2.2) has local minima that are not global minima. An estimate which corresponds to a local minimum will normally not produce

a good model. It is therefore important to take sufficient care before accepting an estimate. For that reason and to speed up convergence a good choice of the initial estimate is important. For a more detailed discussion of non-linear search techniques to minimise the quadratic error function (2.2), see e.g. Vanderplaats [10], Gill et al. [11] and Ljung [2].

2.3.2 Maximum Likelihood Estimation

It was shown in section 2.3.1 that the least squares method is a special case of the prediction error methods (PEM). In connection to PEM it is also important to mention another identification method, namely the maximum likelihood method (ML).

For this purpose a probabilistic model is introduced. A probabilistic model includes both a predictor function and an assumed probability density function for the associated prediction errors $\epsilon(t,\bar{\theta})$. Here, assumed to be a realization of a Gaussian white noise process $\{\mathcal{E}(t)\}$, i.e. $\epsilon(t,\bar{\theta}) = e(t)$ is a sequence of independent and identically distributed random variables of zero mean and variance $\lambda_\mathcal{E}$.

Using maximum likelihood estimation the parameter estimate $\hat{\bar{\theta}}_N$ is obtained by maximizing the likelihood function $L(\bar{\theta},\lambda_\mathcal{E})$, i.e. the probability density function of the observations $y(t)$ conditioned on the parameter vector $\bar{\theta}$. Because of the linear transformation between the observations $y(t)$ and the prediction errors $\epsilon(t,\bar{\theta})$ it is equally valid to use the probability density function of the prediction errors. The likelihood function is thus given by

$$L(\bar{\theta},\lambda_\mathcal{E}) = \prod_{t=1}^{N} f_\mathcal{E}(\epsilon(t,\bar{\theta}),t;\bar{\theta}) = \prod_{t=1}^{N} \frac{1}{\sqrt{2\pi\lambda_\mathcal{E}}} \exp\left(-\frac{1}{2}\frac{\epsilon^2(t,\bar{\theta})}{\lambda_\mathcal{E}}\right) \tag{2.7}$$

where $f_\mathcal{E}(\cdot,\cdot;\cdot)$ is the probability density function of $\{\mathcal{E}(t)\}$. Normally maximum likelihood estimates of $\bar{\theta}$ are obtained from maximum of

$$\log L(\bar{\theta},\lambda_\mathcal{E}) \tag{2.8}$$

Thus the maximum likelihood estimate of $\bar{\theta}$ is obtained by maximizing the log-likelihood function given by

$$\log L(\bar{\theta},\lambda_\mathcal{E}) = -\frac{N}{2} log\lambda_\mathcal{E} - \frac{1}{\lambda_\mathcal{E}}\frac{N}{2}\frac{1}{N}\sum_{t=1}^{N}\epsilon^2(t,\bar{\theta}) + \text{constant} \tag{2.9}$$

which can be written

$$\log L(\bar{\theta},\lambda) = -\frac{N}{2}\left[\frac{R_N(\bar{\theta})}{\lambda_\mathcal{E}} + log\lambda_\mathcal{E}\right] + \text{constant} \tag{2.10}$$

where

$$R_N(\bar{\theta}) = \frac{1}{N}\sum_{t=1}^{N}\epsilon^2(t,\bar{\theta}) \tag{2.11}$$

Differentiation of (2.10) with respect to λ_ε gives

$$\frac{\partial \log L(\bar{\theta}, \lambda_\varepsilon)}{\partial \lambda_\varepsilon} = -\frac{N}{2}\left(-\frac{R_N(\bar{\theta})}{\lambda_\varepsilon^2} + \frac{1}{\lambda_\varepsilon}\right) \tag{2.12}$$

and

$$\frac{\partial^2 \log L(\bar{\theta}, \lambda)}{\partial \lambda_\varepsilon^2} = \frac{N}{2}\left(-\frac{2R_N(\bar{\theta})}{\lambda_\varepsilon^3} + \frac{1}{\lambda_\varepsilon^2}\right) \tag{2.13}$$

From (2.12) and (2.13) the estimate of λ_ε is found to be

$$\hat{\lambda}_N = R_N(\bar{\theta}) \tag{2.14}$$

where $\bar{\theta}$ is to be replaced by its optimal value, which is to be determined. Inserting (2.14) into (2.10) gives

$$\log L(\bar{\theta}, \hat{\lambda}_N) = -\frac{N}{2} log R_N(\bar{\theta}) + \text{constant} \tag{2.15}$$

so the maximum likelihood estimate of $\bar{\theta}$ is obtained by minimizing $R_N(\bar{\theta})$. The minimizing point will be the estimate $\hat{\bar{\theta}}_N$ and the value $R_N(\hat{\bar{\theta}}_N)$ will become the estimate $\hat{\lambda}_N$. So the ML method becomes a PEM provided the prediction errors are Gaussian distributed.

2.4 Asymptotic Theory for the Prediction Error Parameter Estimate

In this section, the question of what would happen to the estimate $\hat{\bar{\theta}}_N$ if more and more observed data become available.

When random disturbances are present minimization of the criterion (2.2) will generally lead to a value of $\hat{\bar{\theta}}_N$ which is realization dependent, i.e. $\hat{\bar{\theta}}_N$ is a random variable. The most important questions to ask of the parameter estimate $\hat{\bar{\theta}}_N$ obtained by the PEM for increasing data sets , i.e. as $N \to \infty$, are

(1) Do the estimates converge ?

(2) In what statistical sense do they converge ?

(3) To what values do they converge ?

(4) What is the asymptotic distribution of the estimates?

From the point of view of engineering application it is important that the estimates converge. Furthermore, it is clear that estimates should converge with probability one (w.p.1), meaning that the value obtained will certainly be close to the best

limiting value. Here, it is not strictly correct to speak of true parameter values and true model as the model chosen will rarely coincide exactly with the system generating the observations. Therefore, the word best is used instead of true.

To answer the four questions, asked above, an asymptotic theory for the estimate $\hat{\bar{\theta}}_N$ may be developed, see e.g. Ljung et al. [12], Ljung [2], Söderström et al. [4] and Goodwin et al. [1]. Here some important basic results given in these references are quoted.

2.4.1 Asymptotic Estimates

It can be shown under quite general conditions that

$$\hat{\bar{\theta}}_N \to \bar{\theta}^* \quad w.p.1 \quad as \quad N \to \infty \tag{2.16}$$

where the unique minimum or the limiting value $\bar{\theta}^*$ of a given model $\mathcal{M}(\bar{\theta})$ is given by

$$\bar{\theta}^* \triangleq \arg\min_{\bar{\theta}} \bar{V}(\bar{\theta}) = \arg\min_{\bar{\theta}} \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E[\epsilon^2(t, \bar{\theta})] \tag{2.17}$$

$E[\cdot]$ is the expectation operator. It can be shown that (2.16) holds even if the system cannot be exactly represented within the model structure $\mathcal{M}$, $\mathcal{S} \notin \mathcal{M}$. I.e. $\mathcal{M}(\bar{\theta}^*)$ can be interpreted as the best approximation of the system $\mathcal{S}$ that is available in the model structure $\mathcal{M}$.

If $\mathcal{S} \in \mathcal{M}$

$$\bar{\theta}^* = \bar{\theta}_0 \tag{2.18}$$

where $\bar{\theta}_0$ is the true parameter estimate. $\bar{\theta}_0$ gives a correct description of the system, i.e.

$$\mathcal{S} = \mathcal{M}(\bar{\theta}_0) \tag{2.19}$$

provided the data set Z^N is informative enough. According to section 2.2.2 the data set is capable of distinguishing between different models.

The above result shows that the PEM estimate $\hat{\bar{\theta}}_N$ is consistent, i.e $\hat{\bar{\theta}}_N$ converges to $\bar{\theta}_0$.

2.4.2 Asymptotic Distribution

Following the discussion of the limit of the PEM estimate $\hat{\bar{\theta}}_N$, this section examines the limiting distribution.

It can be shown that the prediction error estimate $\hat{\bar{\theta}}_N$ is asymptotically normally distributed. Under the assumption of stationarity the distribution is given by

$$\sqrt{N}(\hat{\bar{\theta}}_N - \bar{\theta}^*) \xrightarrow{dist} N(0, \bar{\bar{P}}_{\hat{\bar{\theta}}_N}) \tag{2.20}$$

where *dist* is an abbreviation for distribution. $\overline{\overline{P}}_{\hat{\theta}_N}$ is an asymptotic covariance matrix.

(2.20) holds, like (2.16), even if the system $\mathcal{S}$ cannot be exactly represented within the set of models $\mathcal{M}$, $\mathcal{S} \notin \mathcal{M}$.

Further, it can be shown that (2.20) implies that the parameter covariance matrix $\overline{\overline{C}}_{\hat{\theta}_N}$ of the PEM estimate $\hat{\bar{\theta}}_N$ is given

$$\overline{\overline{C}}_{\hat{\theta}_N} \approx \frac{1}{N}\overline{\overline{P}}_{\hat{\theta}_N} \tag{2.21}$$

(2.21) gives a means of determining the accuracy of the PEM estimate $\hat{\bar{\theta}}_N$. This is very important since the estimate by itself is of little value without a measure of its accuracy. Further (2.20) shows that the same asymptotic covariance is achieved whenever a PEM is used, whether or not the prediction errors are Gaussian.

2.4.3 Asymptotic Covariance Matrix

Expressions for the asymptotic covariance matrix $\overline{\overline{P}}_{\hat{\theta}_N}$ have been developed under varying assumptions, see e.g. Ljung [2] and Söderström [4]. It should be noted that it is more unmanageable to estimate $\overline{\overline{P}}_{\hat{\theta}_N}$ in the more general situation $\mathcal{S} \notin \mathcal{M}$ than in the situation when $\mathcal{S} \in \mathcal{M}$. Here, the result for $\mathcal{S} \in \mathcal{M}$ will be given.

Assume $\mathcal{S} \in \mathcal{M}$ such that $\bar{\theta}^* = \bar{\theta}_0$ will minimise the prediction error $\epsilon(t, \bar{\theta}_0) = e(t)$ where $e(t)$ is a sequence of independent random variables with zero mean values and variances λ_ε. Then the asymptotic covariance matrix is given by

$$\overline{\overline{P}}_{\hat{\theta}_N} = \lambda_\varepsilon \left(\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E[\overline{\Psi}(t,\bar{\theta}_0)\overline{\Psi}^T(t,\bar{\theta}_0)] \right)^{-1} = \lambda_\varepsilon \left(\overline{E}[\overline{\Psi}(t,\bar{\theta}_0)\overline{\Psi}^T(t,\bar{\theta}_0)] \right)^{-1} \tag{2.22}$$

where a realization of the stochastic process $\{\overline{\Psi}(t,\bar{\theta})\}$ is given by

$$\overline{\psi}(t,\bar{\theta}_0) \triangleq -\frac{d\epsilon(t,\bar{\theta})}{d\bar{\theta}}\Big|_{\bar{\theta}=\bar{\theta}_0} = \frac{d\hat{y}(t|\bar{\theta})}{d\bar{\theta}}\Big|_{\bar{\theta}=\bar{\theta}_0} \tag{2.23}$$

$\overline{\psi}(t,\bar{\theta}_0)$ is the gradient of $\hat{y}(t|\bar{\theta}_0)$. $\overline{\psi}(t,\bar{\theta}_0)$ is a column vector. It may be noticed that an expression corresponding to (2.22) also exists for a MIMO system.

From (2.22) it is seen that the asymptotic accuracy of the prediction error estimate $\hat{\bar{\theta}}_N$ can be estimated from the data set Z^N if $\bar{\theta}_0$, λ_ε and $\overline{E}$ are known. However, it can be shown that reasonable estimates $\hat{\overline{\overline{P}}}_N$ of $\overline{\overline{P}}_{\hat{\theta}_N}$ can be obtained by replacing $\bar{\theta}_0$ by $\hat{\bar{\theta}}_N$ and $\overline{E}$ by the sample sum. Therefore, an estimate of (2.22) can be written

$$\hat{\overline{\overline{P}}}_N = \hat{\lambda}_N \left(\frac{1}{N} \sum_{t=1}^{N} \overline{\psi}(t,\hat{\bar{\theta}}_N)\overline{\psi}^T(t,\hat{\bar{\theta}}_N) \right)^{-1} \tag{2.24}$$

where

$$\hat{\lambda}_N = \frac{1}{N}\sum_{t=1}^{N}\epsilon^2(t,\hat{\bar{\theta}}_N) \tag{2.25}$$

It is important to notice that the parameter covariance matrix can be estimated by using (2.24)-(2.25) if just one realization of the response is obtained.

Investigating (2.24) it is seen that the asymptotic accuracy of a certain parameter is related to the sensitivity of the prediction error $\epsilon(t,\hat{\bar{\theta}}_N)$ with respect to this parameter. It is also seen that the accuracy depends on the prediction error variance $\hat{\lambda}_N$ and the number of data N. If the prediction errors are Gaussian distributed white noise it also can be seen from (2.24) that the PEM estimate $\hat{\bar{\theta}}_N$ is asymptotically efficient, i.e. has a minimum possible variance. This means that the covariance matrix $\overline{\overline{C}}_{\hat{\bar{\theta}}_N}$ of the PEM estimate (under the Gaussian assumption equal ML) is equal to the Cramér-Rao lower bound for the covariance matrix of an unbiased estimator. It can be shown, see e.g. Ljung [2] that the covariance of an unbiased estimator $\hat{\bar{\theta}}_N$ satisfies the Cramér-Rao inequality

$$\overline{\overline{C}}_{\hat{\bar{\theta}}_N} \geq \overline{\overline{J}}^{-1} \tag{2.26}$$

where the matrix $\overline{\overline{J}}$ is known as the Fisher information matrix

$$\overline{\overline{J}} = E_{Y|\bar{\theta}_0}\left[\left(\frac{\partial log L(\bar{\theta},\lambda_\varepsilon)}{\partial\bar{\theta}}\right)\left(\frac{\partial log L(\bar{\theta},\lambda_\varepsilon)}{\partial\bar{\theta}}\right)^T\right] \tag{2.27}$$

$L(\bar{\theta},\lambda_\varepsilon)$ is the likelihood function. If a probability model is available and the measured response $y(t)$ of the system can be described by

$$y(t) = y(t|\bar{\theta}) + e(t) \tag{2.28}$$

where the additive noise $e(t)$ is a white noise sequence having a Gaussian distribution with zero mean and variance λ_ε. Here (2.9) implies that

$$\frac{\partial log L(\bar{\theta},\lambda_\varepsilon)}{\partial\bar{\theta}} = -\frac{1}{\lambda_\varepsilon}\sum_{t=1}^{N}\epsilon(t,\bar{\theta})\bar{\psi}(t,\bar{\theta}) \tag{2.29}$$

Using the fact that $e(t)$ is a realization of a white noise process with zero mean and variance λ_ε (2.27) implies that the Cramér-Rao inequality for the PEM estimate $\hat{\bar{\theta}}_N$ can be written

$$\overline{\overline{C}}_{\hat{\bar{\theta}}_N} \geq \lambda_\varepsilon\left(\sum_{t=1}^{N}E[\bar{\Psi}(t,\bar{\theta})\bar{\Psi}^T(t,\bar{\theta})]\right)^{-1} \tag{2.30}$$

Combining (2.21) and (2.22) it is seen that the result per sample for $N \to \infty$ gives the limit of the Cramér-Rao lower bound in case the prediction errors are

Gaussian distributed. This means, as mentioned above, that the estimate $\hat{\bar{\theta}}_N$ is asymptotically efficient.

If the estimate $\hat{\bar{\theta}}_N$ is biased, result similar to (2.26) applies. Assume that

$$E[\hat{\bar{\theta}}_N] = \gamma(\bar{\theta}) \tag{2.31}$$

then

$$\bar{\bar{C}}_{\hat{\bar{\theta}}_N} \geq \left(\frac{\partial\gamma(\bar{\theta})}{\partial\bar{\theta}}\right)\bar{\bar{J}}^{-1}\left(\frac{\partial\gamma(\bar{\theta})}{\partial\bar{\theta}}\right)^T \tag{2.32}$$

It may be noticed that (2.24) gives the asymptotic covariance provided

- a PEM is used with a quadratic criterion function (2.2) and
- $\bar{\theta}^* = \bar{\theta}_0$ such that $\epsilon(t, \bar{\theta}_0)$ is a realization of a white noise process with zero mean and variance λ_ε.

On the other hand, (2.26) holds for any unbiased estimator. The expression (2.30) requires that the noise in the model (2.28) is additive and Gaussian. If the noise is not additive or Gaussian the covariance matrix can still be obtained from (2.26) as long as an efficient estimator is used. Further, (2.26) is available if the processes are not stationary contrary to (2.24) that is valid only for stationary processes. However, it may be noticed that it can be cumbersome to estimate the Fisher Information matrix in the general case because of the amount of work entailed in calculating the likelihood function and then taking the expectation.

It is seen from (2.24) and (2.30) that a small variance in a certain parameter θ_i results if the predictor $\hat{y}(t|\bar{\theta})$ is sensitive to those components. This implies that an advice for the choice of experimental conditions $\mathcal{H}$ could be to choose the experimental conditions so that the predicted output becomes sensitive with respect to parameters that are important for the application in question. Further, it is seen from the above that experiment design based on (2.24) and (2.30) is equivalent if the noise is additive and Gaussian white. In the following chapters the problem of the choice of experimental conditions will be discussed further.

2.5 Example 2.1: Estimation of the Parameter Covariance Matrix

It may be noticed that (2.24) is an asymptotic covariance in N, the number of observed data. The theory behind the expression does not tell how large N has to be for the results to be applicable. In the following example it is investigated whether the expression for the asymptotic covariance is relevant for finite N by comparing it with the sample covariance matrix obtained by performing simulations.

The measurements $y(t)$ are assumed to be given by

$$y(t) = y(t|\bar{\theta}) + e(t) \tag{2.33}$$

It is assumed that the system can be described by

$$y(t|\bar{\theta}) = \theta_1 \sin(\theta_2 t + \theta_3) + \theta_4 \sin(\theta_5 t + \theta_6) \tag{2.34}$$

$e(t)$ is a realization of a Gaussian stochastic noise process $\{\mathcal{E}(t)\}$ with the variance $\lambda_{\mathcal{E}}$.

For each simulation run, the parameter estimates $\hat{\bar{\theta}}_N$ were determined using the least-squares method. The estimate in run i is denoted $\hat{\bar{\theta}}_N^{(i)}$. The sample covariance matrix is then given by

$$\hat{\bar{C}}_{\hat{\bar{\theta}}_N} = \frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} (\hat{\bar{\theta}}_N^{(i)} - \mu_{\hat{\bar{\theta}}_N})(\hat{\bar{\theta}}_N^{(i)} - \mu_{\hat{\bar{\theta}}_N})^T \tag{2.35}$$

where the sample average of the parameter estimates $\mu_{\hat{\bar{\theta}}_N}$ is given by

$$\mu_{\hat{\bar{\theta}}_N} = \frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} \hat{\bar{\theta}}_N^{(i)} \tag{2.36}$$

N_{sim} is the number of simulations. The system was simulated for $t = 1, 2, .., N$ where N is the number of data points.

The simulations were performed by using the following data

- $\lambda_{\mathcal{E}} = 0.05$
- N=100,N=500
- $N_{sim} = 500$
- $\bar{\theta}^T = \{\theta_1, \theta_2, .., \theta_6\} = \{1\,2\,3\,4\,5\,6\}$

The parameter covariance matrix estimated by (2.24) is given by

$$\hat{\bar{C}}_{\hat{\bar{\theta}}_N} = \frac{1}{N10} \begin{pmatrix} +0.0525 & +0.0004 & -0.0041 & -0.0025 & -0.0001 & +0.0009 \\ +0.0004 & +0.0057 & -0.0292 & +0.0019 & -0.0001 & -0.0002 \\ -0.0041 & -0.0292 & +0.1989 & +0.0116 & +0.0002 & +0.0001 \\ -0.0025 & +0.0019 & +0.0116 & +0.0516 & +0.0002 & -0.0008 \\ -0.0001 & -0.0001 & +0.0002 & +0.0002 & +0.0040 & -0.0020 \\ +0.0009 & -0.0002 & +0.0001 & -0.0008 & -0.0020 & +0.0130 \end{pmatrix} \tag{2.37}$$

The estimated sample covariance matrix estimated by (2.35) for

$N = 100$ is given by

$$\hat{\hat{\overline{C}}}_{\hat{\underline{\theta}}_N} = \frac{1}{1000}\begin{pmatrix} +0.0500 & +0.0004 & -0.0015 & -0.0061 & -0.0000 & +0.0006 \\ +0.0004 & +0.0061 & -0.0304 & +0.0008 & -0.0001 & -0.0001 \\ -0.0015 & -0.0304 & +0.2033 & +0.0048 & +0.0008 & +0.0006 \\ -0.0061 & +0.0008 & +0.0048 & +0.0425 & +0.0001 & -0.0015 \\ -0.0000 & -0.0001 & +0.0008 & +0.0001 & +0.0004 & -0.0022 \\ +0.0006 & -0.0001 & +0.0006 & -0.0015 & -0.0022 & +0.0152 \end{pmatrix} \tag{2.38}$$

where the corresponding estimated parameter vector is

$$\mu_{\hat{\underline{\theta}}_N}^T = \{0.9958\, 2.0010\, 2.9914\, 3.9979\, 4.9911\, 6.0056\} \tag{2.39}$$

The estimated sample covariance matrix for
$N = 500$ is given by

$$\hat{\hat{\overline{C}}}_{\hat{\underline{\theta}}_N} = \frac{1}{5000}\begin{pmatrix} +0.0512 & +0.0004 & -0.0033 & -0.0050 & -0.0000 & +0.0008 \\ +0.0004 & +0.0060 & -0.0290 & +0.0015 & -0.0001 & -0.0001 \\ -0.0033 & -0.0290 & +0.2003 & +0.0048 & +0.0001 & -0.0001 \\ -0.0050 & +0.0015 & +0.0048 & +0.0485 & +0.0001 & -0.0010 \\ -0.0000 & -0.0001 & +0.0001 & +0.0001 & +0.0034 & +0.0019 \\ +0.0008 & -0.0001 & -0.0001 & -0.0010 & +0.0019 & +0.0147 \end{pmatrix} \tag{2.40}$$

where the corresponding estimated parameter vector is

$$\mu_{\hat{\underline{\theta}}_N}^T = \{1.0000\, 2.0001\, 2.9997\, 3.9999\, 4.9981\, 6.0007\} \tag{2.41}$$

By comparing (2.37) with (2.38) and (2.40) it is seen that the theoretical values give a good indication for what is to be expected in the practice. Further, by estimating the sample mean and sample standard deviation over the 500 simulations for $\hat{\hat{\overline{C}}}_{\hat{\underline{\theta}}_N}$ gave results that showed a good agreement between the simulated and theoretical values. This means that the asymptotic expression (2.24) is also applicable for finite N. However, it may be noticed that the values in (2.37) correspond to one realization.

2.6 Bayesian Approach to Parameter Estimation

The parameter estimators considered above were motivated by the simple idea of fitting model output to observed output as closely as possible. They proved to have attractive statistical properties under suitable assumptions. However, following questions about them remain unanswered

- How can a priori information of the most likely values for the parameters be taken into account ?
- Regarding the parameters as random variables can their probability density function (p.d.f.) rather than just their means be estimated. The p.d.f. says much more about the parameters than a point estimate and its covariance.

The aim of this section is to answer these questions by describing a so-called Bayesian approach for parameter estimation, see e.g. Beliveau [13], Peterka [14] or Beck et al. [15]. It may be noticed that in a Bayesian approach to parameter estimation the parameters $\overline{\theta}$ are assumed to be random variables while in the foregoing sections the parameter estimate $\hat{\overline{\theta}}_N$ is assumed to be random. In Bayesian statistics the unknown parameters $\overline{\theta}$ are actually not estimated but the posterior probability density function for them is calculated. Calculation of the posterior probability density function is called "Bayesian estimation" .

2.6.1 Bayesian Estimation

Once the measurements consisting of a batch Y^N of N measured data have been obtained from the system to be identified the posterior knowledge of $\overline{\theta}$ has the conditional posterior p.d.f. $f(\overline{\theta}|Y^N)$ given by Bayes' rule

$$f(\overline{\theta}|Y^N) = \frac{f(Y^N|\overline{\theta})f(\overline{\theta})}{f(Y^N)} \tag{2.42}$$

where $f(\overline{\theta})$ is the prior p.d.f. of $\overline{\theta}$. $f(\overline{\theta})$ reflects the user's prior confidence provided from background knowledge and assumption. $f(Y^N|\overline{\theta})$ is the likelihood function reflecting all what the experiment can say about the unknown parameters.

One of the most valuable features of Bayesian estimation is its applicability for estimation in steps, bringing new information at each step. Bayes estimation in steps involves use of the posterior p.d.f. from each step as the prior p.d.f. for the next. If a collection of data Y^N is used in conjunction with an initial distribution to get an improved distribution by (2.42) and this posterior distribution is used as a prior distribution with a second collection of data, then the obtained second posterior distribution is the same as the distribution that would be obtained by using a combination of the two collections of data with the initial a prior distribution, see e.g. Ditlevsen [16].

Having recognised the attractions of Bayes estimation its drawbacks will now be mentioned.

The need to provide a prior p.d.f. is hardest to meet; indeed some statisticians find themselves unable to do so with clear consequence, because it is subjective. The prior p.d.f may be interpreted as starting degrees of the belief. A further drawback is that the information included in the measurements far overweight that contained in the prior p.d.f so the final result is not very sensitive to the prior

p.d.f. Another fundamental drawback of Bayesian estimation is the amount of work entailed in calculating the posterior p.d.f. and then extracting a parameter estimate. This problem is considered in the following section.

2.6.2 Bayesian Estimators

From the posterior p.d.f. different estimates of the parameter vector $\overline{\theta}$ can be obtained. A Bayesian point estimator $\hat{\overline{\theta}}$ of the parameter vector $\overline{\theta}$ may be determined such that the expected loss associated with the prediction error between $\hat{\overline{\theta}}$ and $\overline{\theta}$ is minimized, see e.g. Norton [3].

The Bayes estimator $\hat{\overline{\theta}}$ minimizing the expected value of a loss function $g_l(\overline{\theta}, \hat{\overline{\theta}})$ over all possible $\overline{\theta}$ given the measurements Y^N is found from

$$\frac{\partial}{\partial \hat{\overline{\theta}}} \left\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g_l(\overline{\theta}, \hat{\overline{\theta}}) f(\overline{\theta}|Y^N) d\overline{\theta} \right\} = 0 \tag{2.43}$$

If a quadratic loss function

$$g_l(\overline{\theta}, \hat{\overline{\theta}}) = (\overline{\theta} - \hat{\overline{\theta}})^2 \tag{2.44}$$

is used it is easy to see that the Bayes estimate $\hat{\overline{\theta}}$ is the posterior mean value of $\overline{\theta}$. In the same way it is easy to see that $\hat{\overline{\theta}}$ is the median of $\overline{\theta}$ if the loss function is linear. The value of $\overline{\theta}$ for which the posterior p.d.f. attains its maximum is known as the maximum posterior estimate (MAP) of $\overline{\theta}$. If the prior information has an insignificant influence the MAP estimate is close to the maximum likelihood estimate. For a symmetric posterior p.d.f. that has only one maximum the different estimates (mean, median, MAP) yield the same result, see e.g. Norton [3]. This means that the three estimates coincide when $\overline{\theta}$ is posterior Gaussian distributed.

If a quadratic loss function is used and the noise is assumed to be Gaussian it is shown in Beliveau [13] that the mean of $\hat{\overline{\theta}}$ can be estimated by minimizing the following loss function

$$V_N(\overline{\theta}, Z^N) = \overline{\epsilon}^T \overline{\overline{C}}_{\varepsilon\varepsilon}^{-1} \overline{\epsilon} + (\overline{\theta} - E[\overline{\theta}_p])^T \overline{\overline{C}}_{\hat{\overline{\theta}}_p}^{-1} (\overline{\theta} - E[\overline{\theta}_p]) \tag{2.45}$$

and the covariance $\overline{\overline{C}}_{\hat{\overline{\theta}}}$ of the parameter estimate $\hat{\overline{\theta}}$ is given by

$$\overline{\overline{C}}_{\hat{\overline{\theta}}} = \left(\overline{\overline{C}}_{\hat{\overline{\theta}}_p}^{-1} + \overline{\overline{\Delta}}^T \overline{\overline{C}}_{\varepsilon\varepsilon}^{-1} \overline{\overline{\Delta}} \right)^{-1} \tag{2.46}$$

where $\overline{\overline{C}}_{\varepsilon\varepsilon}$ is the covariance matrix of the noise process, $\overline{\overline{C}}_{\hat{\overline{\theta}}_p}$ is the priori covariance and $E[\overline{\theta}_p]$ is the prior mean of $\overline{\theta}$. $\overline{\epsilon}$ is a N-dimensional vector containing the prediction errors $\epsilon(t, \overline{\theta})$ and $\overline{\overline{\Delta}}$ is a matrix containing the gradients $\psi(t, \hat{\overline{\theta}})$.

2.7 Model Structure Selection and Model Validation

In section 2.2.2 it is explained that in system identification both the determination of the model structure $\mathcal{M}$ and model validation are important for a successful identification application. E.g. an overparameterized model structure, i.e. containing several models giving a perfect description of the system, can lead to unnecessarily complicated computations for finding the parameter estimates. An underparametrized model, i.e. a model having few parameters to describe the system adequately, may be inaccurate.

The purpose of this section is to quote some basic methods that can be used to find an appropriate PEM model structure and to validate the best model pointed out within the chosen structure.

2.7.1 Choice of Model Structure

In section 2.1 a list of typical model structures to be used for system identification was given. In the identification procedure the user has to choose a model structure and then make an appropriate choice within this structure for a best model. E.g. the general SISO-model is an example of a model structure.

The choice of the model structure involves:

- *Model type.* This involves the selection between non- linear and linear models, between black-box and physical models etc..
- *Model size.* I.e. choice of model order and number of adjustable parameters.
- *Model parametrization.* I.e. the way in which the number of parameters enter into the model.

The choice of the model structures to a large extent should be made according to the aim of the final purpose. There are indeed many other factors that influence the selection of the model structure. Four of the most important factors are:

- *Flexibility.*
- *Parsimony.*
- *Algorithm complexity.*
- *Properties of the criterion function (2.2).*

There is one aspect related to the four factors that should be analysed here. This concerns the problem of constraining the model structure in consequence of the parsimony principle. The principle says that out of two or more competing models which all explain the data well, the model with the smallest number of independent

parameters should be chosen. This rule is quite in line with common sense: "Do not use extra parameters for describing a dynamic phenomenon if they are not needed".

The parsimony principle can be illustrated by an expression for the expected prediction error variance where the expectation is with respect to $\hat{\bar{\theta}}_N$, see e.g. Söderström [4]

$$E[W_N(\hat{\bar{\theta}}_N)] \approx \lambda_\varepsilon \left(1 + \frac{\dim\bar{\theta}}{N}\right) \tag{2.47}$$

where $\dim\bar{\theta}$ is the dimension of the parameter vector $\bar{\theta}$ and

$$W_N(\hat{\bar{\theta}}_N) = E[\epsilon^2(t, \bar{\theta}_N)] \tag{2.48}$$

is the prediction error variance when the model corresponding to $\hat{\bar{\theta}}_N$ is used.

It is seen that this measure of accuracy depends only on the number of data N and the number of parameters in $\bar{\theta}$. The model structure $\mathcal{M}$ and the experimental conditions $\mathcal{H}$ do not affect the result contrary to the asymptotic expression for the accuracy of $\bar{\theta}$ (2.22) which is related to the particular model $\mathcal{M}(\bar{\theta})$ and the experimental conditions. I.e. sometimes it is more relevant to consider (2.47) instead of (2.22) for the accuracy of a system identification problem.

Further, (2.46) says that the expected prediction error variance $E[W_N(\hat{\bar{\theta}}_N)]$ increases with a relative amount $\frac{\dim\bar{\theta}}{N}$. Thus there is a penalty in using models with an unnecessarily high numbers of parameters. This is in conflict with the fact that flexible models are needed to ensure small bias. So clearly a compromise has to be found. Increasing the number of parameters will reduce the bias and increase the variance. The question how to reach a sensible compromise is a heartpoint of the system identification process. I.e. the key issue is to find flexible models that are described with few parameters. There is no general solution of this problem but a large number of methods to assist in the choice of an appropriate model structure exists. These methods can be divided into several categories. They are based on

- *A priori knowledge.* Information about the system obtained from e.g. understanding of the physics of the system, design calculations, etc.
- *Preliminary data analysis.* Extracting information from the data that involve determination of a complete model of the system. E.g. spectral analysis estimates will give valuable information about resonance peaks. Further, a preliminary data analysis test for non-linear effects can be performed.
- *Comparison of model structures.* A most natural approach to search for a suitable model structure is simply to test a number of different ones and then to compare the resulting models. However, it is usually only feasible to do this with simple models because of the amount of calculation involved in more complicated models.

For such comparisons, as mentioned above a discriminating criterion is needed. The comparison of the model structures can be interpreted as a test for a significant decrease in the minimal values of the loss function $V_N(\bar{\theta}, Z^N)$ associated with the model structures in question. As a model structure is expanded, e.g. increasing the number of adjustable parameters, the minimal value of $V_N(\bar{\theta}, Z^N)$ decreases since new degrees of freedom have been added to the optimization problem. The decrease of $V_N(\bar{\theta}, Z^N)$ is a consequence that more flexible model structures give a possibility for better fit to the data. On the other hand when a good fit can be obtained there is no reason to increase e.g. the number of adjustable parameters. In fact it follows from (2.46) that there is an inherent penalty for a too flexible model structure. Considerations of this type led Akaike, see Akaike [17], to suggest a criterion of the type

$$\bar{\theta}_N^{AIC} = \arg\min_{\bar{\theta}} \left\{ V_N(\bar{\theta}, Z^N) + \frac{2\dim\bar{\theta}}{N} \right\} \tag{2.48}$$

for the determination of the model structure and the parameter estimates. The model structure giving the smallest value of this criterion Akaike's information criterion (AIC) is selected as (2.48) decides whether a decrease of the prediction error criterion $V_N(\bar{\theta}, Z^N)$ corresponding to an increase of the model structure is significant or not.

In e.g. Ljung [2] and Söderström [4] other approaches to model structure comparisons are given.

2.7.2 Model Validation

Model validation is the final stage of the system identification procedure. In fact model validation overlaps with model structure selection. Since system identification is an iterative process various stages will not be separated: models are estimated and the validation results will lead to new models etc.

Model validation involves two basic questions:

- What is the best model within the chosen model structure ?
- Is the model fit for its purpose ?

One of the dilemmas in model validation is that there are many different ways to determine and compare the quality of the estimated models. First of all, the subjective judgement in model validation should be stressed. It is the user that makes the decision based on numerical indicators.

The variance of the parameter estimates can be such an indicator. High values indicate a model with a bad fit or overparameterization.

It is also important to check whether the model is a good fit for the data recording to which it was estimated. If it is a bad fit it may e.g. indicate that the model represents a local minimum.

Simulation of the system with the actual input and comparing the measured output with the simulated model output can also be used for model validation.

Statistical tests of the prediction errors $\epsilon(t, \overline{\theta})$ are also typically used numerical indicators for model validation. If the statistical distribution of $\epsilon(t, \overline{\theta})$ matches the assumed distribution then it can be concluded that the system dynamics is indeed well represented by the model. Any different trend in the statistical characteristics originally assumed is an indication that either the model or the noise is incorrectly assumed or that the parameters are incorrectly estimated.

The above-mentioned tools for model validation lead to a conclusion as to whether the model is fit for its purpose.

2.8 Summary

Above the basic techniques of system identification for parameter estimation of dynamic systems have been outlined with emphasis on the approaches for estimating the parameter covariance matrix which is the basis for experiment design. The outline has mainly been concerned with parameter estimation based on discrete in time linear SISO models. However, many concepts mentioned in this chapter apply equally if parameter estimation is based on e.g. MIMO models, frequency models, etc.

The characteristics features of parametric estimation based on a linear SISO model can be quoted in the following way:

- A parametric time domain estimation method is based on a model structure $\mathcal{M}$. $\mathcal{M}$ describes a set of models $\mathcal{M}^*$ within which the best is sought for. The set is supposed to be parameterised by a finite-dimensional vector $\overline{\theta}$. An individual model within the set is denoted by $\mathcal{M}(\overline{\theta})$. The search within the model structure $\mathcal{M}$ for the model $\mathcal{M}(\hat{\overline{\theta}}_N)$ that best describes the data Z^N becomes a problem of estimating $\hat{\overline{\theta}}_N$ which is an estimate of the parameter vector $\overline{\theta}$. The data set Z^N will depend on the system $\mathcal{S}$ that has generated the data and the experimental conditions $\mathcal{H}$ that were at hand when the data were recorded.
- The problem, identifiability, whether the identification procedure will yield a unique value of the parameter vector $\overline{\theta}$, and/or whether the resulting model is equal to the true system is central in parameter estimation problems.
- Parameter estimates obtained by prediction error methods (PEM) are asymptotically Gaussian distributed as the number of observed data tends to infinity.
- The PEM is applicable to general model structures and gives optimal asymptotic accuracy when the true system can be represented within the model structure $\mathcal{S} \in \mathcal{M}$ and reasonable approximation properties, when the true system cannot be represented within the model structure.
- The asymptotic covariance matrix for the estimates of the parameters, when

$S \in \mathcal{M}$, is given by the inverse of the covariance matrix of the predictor gradients, normalised by the predictor error variance divided by the number of observed data. The prediction errors are realizations of a white noise process. The expression of the covariance matrix is also the limit of the Cramér-Rao lower bound, in case the predictor errors are Gaussian distributed. This means that the parameter estimates are then asymptotically efficient.

- The expression for the asymptotic covariance of the parameter estimates seems also applicable to a finite number of data.
- In Bayesian estimation it is possible to bring in a priori information of the parameters.

2.9 References

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Chapter 3

Application of System Identification in Civil Engineering

System identification techniques, as describe in the foregoing chapter, have been widely used in many branches of science and engineering for the estimation of various characteristics of a physical system, see e.g. Eykoff [1]. Applications of system identification in civil engineering have been studied with increasing interest during the last two decades. Primarily motivated by the desire to have a more accurate description of the structure and its dynamic characteristics for the purposes of predicting its response to environmental excitations such as earthquake, wave and wind generated pressure loads. Further, to assess safety or damage through changes in structural parameters and finally for the purposes of applying controllers to structures that can reduce environmental excitations. Certainly, for these reasons and, perhaps, others that one may think of, an accurate model of the real structure must be available to the engineer. In general, it can be stated that the aims of structural identification are several where models and techniques differ depending on the aims. Different aims of structural identification are discussed in e.g. Natke et al. [2], Natke [3], Hart et al. [4], Ibanez [5], Aktan et al. [6] and Vestroni et al. [7].

The aim of this chapter is to introduce the type of system identification methods normally used in parametric identification of civil engineering structures. It is not the intent to get a complete outline of different system identification methods used in civil engineering, but only the most fundamental methods. A more thorough outline can e.g. be found in recent survey papers such as Kozin et al. [8]. Comprehensive surveys showing the trend in structural identification are also given in e.g. the following Ph.D. theses: Ahmadi [9], Jayakumar [10], Sprandel [11] and Jensen [12]. Further, it is possible to find the latest developments in system identification techniques, used in civil engineering problems, in the proceedings of national and international conferences such as the IMAC (International Modal Analysis Conference), and the annual seminars in Hannover and Leuven.

In section 3.1 general remarks concerning application of system identification in civil engineering are given. Next, in sections 3.2 and 3.3 the models and excitations, respectively, normally used in system identification of structural systems are described. Then, in section 3.4, frequency domain and time domain system identification methods are presented.

3.1 General Remarks

Structures can be regarded as distributed parameter systems characterized by distribution of the mass, damping and stiffness properties. However, parameter identification in distributed parameter systems is generally not easy. Thus, with a few exceptions, in most of the literature on testing of structures, the data are analysed based on the assumption that the system is described by one or a set of linear ordinary second-order differential equations. Because of their simplicity the linear time invariant lumped parameter models are the most widely used models in structural identification. More complex models such as the linear continuous parameter models and non-linear models are used only when the lumped-parameter model cannot be used to provide an adequate representation of the structural behaviour.

Since a model is chosen a priori in structural system identification the system identification problem in civil engineering is then reduced to the estimation of the parameters measurements.

In general, the identification techniques determine the so-called modal parameters (modal damping, eigenfrequencies, mode shapes). Therefore, the term "experimental modal analysis", is often used instead of "system identification". However, it may be noticed that the term "experimental modal analysis" has earlier been used for one specific parameter estimation method based on so-called frequency-response functions, where the modal parameters are obtained by a curvefit of the frequency-response function of the model to that of the test data, see e.g. Ewins [13] for details. Nowadays, "experimental modal analysis" is synonymous with "system identification" when modal parameter identification problems are considered.

In contrast to the modal model identification, the determination of the mass, damping and stiffness matrices, or parameterizations of it, are available. Thus, the parameters to be identified consist of physical/geometrical parameters. Identification of physical/geometrical parameters is the most straightforward approach for identification of structural systems. The advantage of identification of those parameters rather than the modal quantities is that the engineer may have some a priori knowledge about the physical/geometrical parameters. The disadvantage of identification based on physical parameters is the number of parameters and fewer flexible models.

In general, it must be said that most system identification techniques currently

in use or that being developed are essentially parameter estimations, i.e. they identify parameters of a priori selected models. Only a few approaches do not seek to determine the parameters of assumed models and may be termed "non-parametric". Their objective is to arrive at a functional representation of the system that is capable of predicting the output, given the input.

Most identification techniques is based on frequency methods, also called Fourier techniques. The frequency domain methods seem to have dominated the theory and practice of system identification up to the sixties. From the end of the sixties and onwards the interest in system identification based on time domain methods has increased and now literature on system identification is very much dominated by time domain methods. System identification based on time domain models and frequency models, respectively, can be considered as two complementary approaches. System identification based on frequency models often gives good insight into the properties of the data and the system. Therefore, a frequency domain approach can be a good first step in the data analysis procedure. Often the intended use of the model as well as the accuracy requirements for parameter estimates motivate the use of a time domain model.

In the following section different frequency domain and time domain methods available for parametric identification in civil engineering will be presented. It may be noticed that the methods commonly used are mainly usable for single-input single output systems. However, considerable work has been directed toward developing identification methods for parametric identification of civil engineering structures using multi-input multi-output techniques, see e.g Kano [14].

Further, system identification can be performed for non-linear models but the effort is an order of magnitude greater than that for linear models. However, in Natke et al. [2] it is pointed out that there is a need for development of "extended system identification" methods for complex and highly non-linear structures. Testing of structural systems often indicates that some non-linearities are nearly always present and should be taken into account.

At last it may be noticed that related to system identification for parametric identification techniques are the techniques where the measured response of a linear structure can be used to estimate the forces acting on the structure thus using the structure as a transducer. This case of identification has been used e.g for evaluation of wind forces on structures, see e.g. Natke [3].

3.2 Structural Modelling In System Identification

Parameter estimation of structures is usually greatly simplified, as mentioned above, by certain assumptions of the structures, e.g. that they are linear in the dynamics, time invariant and that they can be adequately described by a discrete mass-spring-damper model. Experience has led to the following mathematical

model for a structure subjected to mechanical loads

$$\overline{\overline{M}}\ddot{\overline{y}}(t) + \overline{\overline{C}}\dot{\overline{y}}(t) + \overline{\overline{K}}\overline{y}(t) = \overline{u}(t) \tag{3.1}$$

where $\overline{\overline{M}}$, $\overline{\overline{C}}$ and $\overline{\overline{K}}$ are the mass, damping and stiffness matrices. $\overline{y}(t)$ and $\overline{u}(t)$ are the displacement and force vectors at the mass points, respectively. Here, it is assumed that the distributed forces of inertia of the structure can be discretized into n degrees-of-freedom and be given by $\overline{\overline{M}}\ddot{\overline{y}}(t)$. This set of forces of inertia is balanced by a set of linear-elastic restoring forces $\overline{\overline{K}}\overline{y}(t)$, viscous damping $\overline{\overline{C}}\dot{\overline{y}}(t)$ and the external loads $\overline{u}(t)$. Assuming a linear structure means that the response of the structure to any combination of forces, simultaneously applied, is the sum of the individual responses to each of the forces acting alone. This is a good assumption for a variety of structures. The time invariant assumption implies that the parameters are to be determined as constants.

The basic goal of parameter estimation is to find the $\overline{\overline{M}}$, the $\overline{\overline{C}}$ and the $\overline{\overline{K}}$ matrices from measured values of $\overline{y}(t)$ and $\overline{u}(t)$ not necessarily being at the same locations. The external forces may be zero if the structure vibrates freely due to an applied velocity and/or initial displacement. In section 3.3 different kinds of excitations normally used for identification of civil engineering structures will be described. However, as mentioned above, it is the modal parameters and not the physical parameters that are determined by most of the system identification techniques.

The modal parameters are obtained when the original equation (3.1) is transformed into a set of uncoupled equations in terms of transformed coordinates (also called generalised coordinated), see e.g. Thomsen [15].

When the modal shape (eigenfunction) $\overline{\phi}_i$ corresponding to the angular eigenfrequencies (eigenvalues) ω_i determined from the eigenvalue problem

$$(\overline{\overline{K}} - \omega_i^2\overline{\overline{M}})\overline{\phi}_i = \overline{0} \qquad i = 1, 2, .., n \tag{3.2}$$

satisfy the orthogonality relation for the damping matrix

$$\overline{\phi}_i^T\overline{\overline{C}}\,\overline{\phi}_j = 0 \qquad i \neq j \tag{3.3}$$

as well as for the mass and stiffness matrix, the displacements $\overline{y}(t)$ can be written in terms of the modal shapes $\overline{\phi}_i$ and the corresponding modal amplitudes $q_i(t)$

$$\overline{y}(t) = \sum_{i=1}^{n} \overline{\phi}_i q_i(t) \tag{3.4}$$

When (3.4) is inserted into (3.1), the equation of motion is transformed into the n transformed uncoupled equations

$$\ddot{q}_i(t) + 2\omega_i\zeta_i\dot{q}_i(t) + \omega^2 g_i(t) = \frac{p_i(t)}{M_i} \qquad i = 1, 2, .., n \tag{3.5}$$

where $q_i(t)$ and $p_i(t)$ are the modal response and the modal load, respectively of the ith mode. ω_i and ζ_i are the undamped eigenfrequency and the modal damping ratio of the ith mode. M_i are the modal mass.

These quantities follow from the variable transformations are

$$\overline{\phi}_i^T \overline{\overline{K}}\, \overline{\phi}_i = \omega_i^2 M_i \tag{3.6}$$

$$\overline{\phi}_i^T \overline{\overline{C}}\, \overline{\phi}_i = 2\omega_i \zeta_i M_i \tag{3.7}$$

$$\overline{\phi}_i^T \overline{\overline{M}}\, \overline{\phi}_i = M_i \tag{3.8}$$

$$p_i(t) = \overline{\phi}_i^T \overline{u}(t) \tag{3.9}$$

It is seen that (3.5) consists of n independent single-degree-of-freedom systems. This means that the modal analysis method is seen to be very efficient, particularly when the system response can be represented by just a few modes which is normally the case when civil engineering structures are considered. The integration of the uncoupled equation of motion is easy; it can be performed analytically if the generalized force $p_i(t)$ is simple, and numerically if the force is complicated. Further, it may be noticed that proportional damping is assumed above. However, the validity of this assumption will not be discussed here, see e.g. Thomsen [15] and Soni [16].

The solution for $t > 0$ to (3.5) is conveniently expressed in the Duhamel integral form

$$q_i(t) = \int_0^t h_i(t-\tau) p_i(\tau) d\tau \tag{3.10}$$

if for simplicity it is assumed that $q_i(0) = \dot{q}_i(0) = 0$. $h_i(t)$ is the impulse response function given by

$$h_i(t) = \frac{1}{M_i \omega_i \sqrt{1-\zeta_i^2}} \exp(-\omega_i \zeta_i t) \sin(\omega_i \sqrt{1-\zeta_i^2}\, t) \qquad t > 0 \tag{3.11}$$

The structural response also can be estimated in the frequency domain by transforming (3.10) into the frequency domain by the Fourier transform. The Fourier transform of the impulse response function gives the complex frequency response function $H(\omega)$ of the ith by

$$H_i(i\omega) = \frac{1}{M_i(\omega_i^2 - \omega^2 + i2\omega\omega_i\zeta_i)} \qquad i = 1, 2, .., n \tag{3.12}$$

In the case where the load $\overline{u}(t)$ can be modelled as a stationary stochastic process the modal response $q_i(t)$ becomes a stochastic process $\{Q_i(t)\}$ and the so-called cross-covariance function of the modal response $C_{Q_iQ_j}(\tau)$ can be written by using (3.10)

$$C_{Q_iQ_j}(\tau) = E[(Q_i(t_1) - E[Q_i(t_1)])(Q_j(t_2) - E[Q_j(t_2)])]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_i(v_1)h_j(v_2)C_{P_iP_j}(v_1 - v_2 + \tau)dv_1dv_2 \qquad (3.13)$$

where $C_{P_iP_j}(\tau)$ is the cross-covariance of the modal load and $\tau = t_2 - t_1$.

By using the Wiener-Khintchine relation the cross-spectral density function can be obtained as follows

$$S_{Q_iQ_j}(\omega) = H_i^*(\omega)H_j(\omega)S_{P_iP_j}(\omega) \qquad (3.14)$$

where $*$ indicates a complex conjugate. $S_{P_iP_j}(\omega)$ follows from (3.9). The cross-spectral density of the stochastic response process $\{Y_i(t)\}$ follows from (3.4)

$$S_{Y_iY_j}(\omega) = \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\phi_l^i\phi_k^j S_{Q_lQ_k}(\omega) \qquad (3.15)$$

The transfer function of the ith degree of freedom due to excitation of the jth degree of freedom is defined by

$$H_{U_jY_i}(\omega) = \frac{S_{Y_iY_i}(\omega)}{S_{U_jU_j}(\omega)} \qquad (3.16)$$

In the above a brief description of the modal approach has been given, mainly to introduce different terms such as modal parameters, impulse response function, frequency response function etc. For details see e.g. Thomsen [15], Meirovitch [17] and Krenk et al. [18].

When the system is described by a linear MDOF (multi-degrees-of-freedom) model, the number of the degrees-of-freedom n of the model must also be determined as part of the system characteristics, unless this information is known a priori. One approach is to determine this number in an indirect manner, i.e. by trial and error, or by using one of the techniques mentioned in chapter 2, available for determination of the model order in system identification.

3.3 Structural Excitation Signals

From an experimental point of view the simplest approach to measure the dynamic parameters of a civil engineering structure is to detect the response to natural forces such as those caused by wind or waves. Such natural loads are called ambient excitation and the vibrations of the structure caused by them are called ambient vibrations. Another source of dynamic excitation for system identification of civil engineering structures can be external excitation.

3.3.1 Ambient Excitation

The ambient excitation is random in nature therefore it cannot be described by an explicit function with time and its characteristics are described by certain statistical parameters, such as mean and standard deviation. This means that the response of the structure is also random and may also be represented by its statistical characteristics. However, the statistical parameters of the response are different from those of the loading. These differences represent the effect of the structure.

Ambient excitation has been shown to be inexpensive, quick and reliable for testing of civil engineering structures such as buildings and offshore structures, see e.g. Ibánẽz [19], Srinavasan et al. [20], Rubin et al. [21] and Jensen [22]. In Jensen [22] an extensive survey of the available literature concerning full-scale measurements on offshore platforms has been performed. It is found that the typical excitation of offshore platforms for system identification is ambient excitation. Further, Morgan et al. [23] conclude, based on a study of several published results of ambient versus forced vibration tests of high-rise structures in USA, that parameter estimates obtained by ambient excitation are as good as parameter estimates obtained by external excitation.

Because of the nature of dynamic testing under ambient excitation conditions this method has advantages over external excitation. Ambient excitation has a wide frequency range, thus theoretically exciting all relevant modes of a structure. Further, ambient excitation testing does not disrupt the normal functioning of a structure and no excitation equipment is required for ambient testing.

It is a disadvantage of ambient excitation tests that the characteristics of the input dynamic forces on the structure cannot be controlled and measured directly, i.e. the input cannot be quantified as to amplitude, spectral content or points of application to the structure. When ambient excitation is measured the observations of the excitation are often given as e.g. time series of the sea surface elevation if system identification of an offshore structure is considered. From the time series characteristics of the sea states, such as significant wave height and average zero upcrossing period, can be estimated. These parameters can then be used as input to models, wave theories, which have been developed to describe waves either as time series or as spectra. The connection between the theoretical description of the waves and the forces on the structure is established using a load model. Such a model is e.g. the well-known Morison equation. A more thorough discussion of the theory used to establish the connection between observations of the waves and the forces on the structure can be found in e.g. Sarpkaya et al. [24].

Instead of measuring the ambient excitation it may often be assumed that these ambient excitations are white in spectrum, at least on a limited frequency band, i.e. a white noise approximation of the input is used for the identification of the structure under consideration.

One disadvantage of ambient excitation is that the amplitude of the forces is often lower than that desired and therefore of limited use in prediction of the properties of non-linear structures, or following the trend of linear properties that change

with amplitude, e.g. damping.

In any case the ambient response of a structure should always be observed, also if external excitation is applied, at least to provide initial guidance for the system identification of the structure.

3.3.2 External Excitation

A survey of external excitation signals for dynamic system identification of civil engineering structures is given in e.g. Schoukens et al. [25] or Natke et al. [26]. Here the most common signals will be mentioned. The different kinds of external excitation can be divided into the following groups:

- Step and impulse excitation
- Random excitation
- Deterministic excitation.

3.3.2.1 Step and impulse excitation

For dynamic tests of civil engineering structures where transient response is desired it is often useful to apply step relaxation (snapback testing) or impulse load. In snapback testing the structure is preloaded with a measured static force through a cable that is suddenly released causing the structure to undergo free vibrations. Snapback testing is convenient if e.g. high amplitude excitation or low frequency excitation is required. It is also useful when an access for mounting a vibration shaker is limited.

Snapback testing is subject to certain limitations in practice. With this excitation technique in general only the lowest natural modes of the structure are sufficiently excited. This disadvantage can be overcame by repeated excitation applied at various locations.

Besides snapback testing impulsive loads can also be used to provide a transient response. An impulsive loading can be established by e.g. an impact or a hammer excitation. The advantage of a test using impulsive loading is e.g. that it is a fast test method for linear systems. Further, the method only requires minimum equipment. The disadvantages are e.g. poor signal to noise ratio and limited control of frequency content.

3.3.2.2 Deterministic Excitation Signals

Apart from the step and impulse excitation techniques mentioned above the most common deterministic excitation signals are derived from sinusoidal signals. Si-

nusoidal signals can be achieved through the use of e.g. sinusoidal eccentric mass vibrators or hydraulic actuators. Sinusoidal signals can either consist of stepped-sine excitation signals or swept-sine excitation signals. A stepped-sine excitation signal is based on sinusoidal signals of constant frequency. At each incremental frequency the vibrators are held at a constant frequency long enough for all transient effects to decay so that only the steady-state response of the structure is recorded, i.e. vibration tests with stepped-sine excitation are extremely time consuming. In order to diminish this disadvantage swept-sine excitation can be used. During the sweep of the entire frequency range the frequency of the vibrator is gradually but continuously varied and the response is continuously recorded. The sweep rate is kept so slow that the excited structure can reach a quasi-steady state. Since all the excitation energy is at one frequency a high signal-to-noise ratio can be obtained.

3.3.2.3 Random Excitation Signals

Instead of exciting the structure by a harmonic signal consisting only of one frequency a random signal may be used. By using a random signal the structure is excited at a frequency range. This means that the test can be carried out faster with random excitation than with harmonic excitation. However, pure random excitation has one disadvantage, since neither the input nor the response is periodic within the measuring time. This causes the so-called leakage error if a FFT-analysis is used, see e.g. Brigham [27]. The leakage error can be avoided if a periodic random excitation is used.

The above review of different structural excitation signals only gives an introduction. A thorough review can e.g. be found in the literature mentioned above or in the Shock & Vibration Handbook [28]. The review has shown that among the different methods of full-scale structural testing ambient vibration measurements are more common than any other. The superiority of ambient testing is due to the ease of experimentation and limited disturbances to the normal operation of the structure. Since measurements must be taken at many shaker frequencies external excitation test duration is longer than for ambient excitation tests.

3.4 System Identification Techniques

In the following section different frequency and time domain techniques normally used for the parametric identification of civil engineering structures will be presented.

All that will be mentioned here about the non-parametric techniques is that non-parametric identification refers, as mentioned in section 3.1, to functional representation, such as the impulse response function in the time domain and the frequency response function in the frequency domain. It may be noticed that although some direct estimation procedures for the estimation of the impulse response function exist, it is done indirectly by inverse Fourier transformation using estimated frequency response function. For the estimation the frequency response function only needs matrix inversion instead of convolution.

From the above the parameter identification techniques are categorised into physical parameter estimation methods and modal parameter estimation methods.

3.4.1 Physical Parameter Estimation

The most commonly used techniques, see e.g. Chen [29], Creamer [30], Sprandel [31], Jensen [12] and Matzen [32], for estimation of the physical parameters are normally integrating the governing equation of the model (3.1) to obtain the estimated model output from measured input and then use a prediction error criterion to minimize the sum of squares error V_N

$$V_N = \frac{1}{N}\sum_{t=1}^{N} \overline{\epsilon}^T(t,\overline{\theta})\overline{\epsilon}(t,\overline{\theta}) \tag{3.17}$$

where

$$\overline{\epsilon}(t,\overline{\theta}) = \overline{y}(t) - \hat{\overline{y}}(t|\overline{\theta}) \tag{3.18}$$

$\hat{\overline{y}}(t|\overline{\theta})$ is the predicted response obtained from integration of the governing equation that is a function of the physical parameters $\overline{\theta}$.

The integration can be performed analytically if the model and the input are simple, or it can be performed numerically, if suitable numerical integration schemes can be derived. Techniques for obtaining the physical parameters have to be used e.g. when the modal analysis technique is unusable, i.e. when non-linearities have to be taken into account. The direct-integration method suffers from problems of numerical stability and accuracy, particularly for non-linear problems. To reduce the effect of these difficulties, it is necessary to verify that the time step choice will lead to convergence to a correct solution. This is accomplished by means of a stability analysis or through a study of the effect of various time steps on the solution. This verification process may be time-consuming and expensive. It is e.g. a disadvantage experienced in the cited literature for estimating the physical

parameters by direct integration are e.g. that the parameters depend on the initial parameter estimates. Further, the covariance of the parameter estimates can be significantly underestimated if the structural excitation is not correctly measured. Finally, it may be noticed that the time-integration also implies that this form of system identification is slow and quickly increasing when the number of parameters is increasing.

3.4.2 Modal Parameter Estimation

The most frequent way of estimating modal parameters is in the time domain, where the free vibration response can be analysed by the so-called logarithmic decrement method to obtain a damping estimate and the period can be measured to estimate the eigenfrequency. The term called logarithmic decrement δ is the logarithm of the ratio of two successive amplitudes $y(t_i)$ and $y(t_{i+1})$

$$\delta = \ln \frac{y(t_{i+1})}{y(t_i)} \tag{3.19}$$

When the logarithmic decrement is determined the damping ratio ζ can be estimated from

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \tag{3.20}$$

The inverse of the eigenfrequency of the system can be obtained by measuring the duration of two zero-crossing points of the displacement history and by dividing the duration by the number of complete cycles between the points. It may be noticed that a free vibration requires an impulse excitation or a snapback testing which is expensive and often by the owner of the structures considered a risk.

In the frequency domain the eigenfrequency ω_0 can be estimated by direct examination of e.g. the peaks of the response spectra or transfer functions if the Fourier transform is used. If it is assumed that the damping ratio is small, i.e. $\zeta < 0.05$, it is easy to show that the damping ratio is related to the half-power bandwidth $B = \omega_2 - \omega_1$ of the system by the equation

$$\zeta \approx \frac{B}{2\omega_0} \tag{3.21}$$

The frequencies ω_1 and ω_2 are called the half-power points, respectively. Assuming small damping implies that the eigenfrequency is approximately equal to the resonance frequency. B is the width of the peak at $\sqrt{2}$ of its height if a frequency response function is considered. If a response spectrum is considered B is the width of the peak at 0.5 of its peak height. The half-power bandwidth method is unreliable and biased. This is a consequence of the method relying on only three points around a resonance peak.

To get more reliable results it is proposed by Vanmarcke [33] to use the spectral moments λ_i

$$\lambda_i = \int_{\omega_a}^{\omega_b} \omega^i S_{Y_j Y_j}(\omega) d\omega \qquad i = 0, 1, 2, ..n \tag{3.22}$$

instead of the ordinates for estimating the damping ratio and eigenfrequency. ω_a and ω_b define the lower and upper frequency limits of integration, respectively. The spectral moments, particularly the three lowermost, tend to be more stable than spectral ordinates. As such, these parameter estimates are thought to be more reliable than those based on spectral ordinates. The method proposed by Vanmarcke [33] is developed further by Sunder et al. [34]. The conclusion in the cited literature is that the spectral moment method is clearly better than the half-power bandwidth method.

Such direct techniques, as mentioned above, are only one parameter-estimation techniques and may fail to work in many situations; other, more systematic ways of fitting the data, may be needed to estimate the parameters. This is particularly true when the data measurements are encumbered with noise.

A simple curvefit method available for estimating the modal parameters of a single-degree-of-freedom system is the so-called single-degree-of-freedom curve-fit or often the circle-fitting method, see Ewins [13], which is a frequency domain method. The method utilises the fact that when the damping is small the so-called Nyquist plot approximately traces a circular arc around the eigenfrequency. A Nyquist plot is obtained by plotting the real and imaginary part of a transfer function against each other for the given frequency range. For a multi-degree-of freedom system a Nyquist plot can be plotted for each resonance frequency . However, if the modes are clustered the method becomes inadequate since the circle-fit method is based on a single-degree-of-freedom assumption and therefore assumes that near a resonance, the behaviour of most systems is dominated by a single mode. When the circle of each Nyquist plot is determined it is easy to estimate the damping from the plots by simple expressions, see Ewins [13].

The most commonly used curve-fitting method in the frequency domain is as mentioned in section 3.1 a parameter estimation method based on so-called frequency-response functions, where the modal parameters are obtained by a curve-fit of the frequency-response function of the model to that of the test data, see e.g. Ewins [13] for details. In this method the frequency response functions are measured using excitation at single or multiple points.

The methods mentioned above assume that the excitation is measured. In Rytter et al. [35] it is shown that it is possible to estimate the modal parameters by using a global curve-fit of the response spectrum of the model to that of the measured data. The idea in the approach is to make a system identification based only on measurements of the response and not of the excitation. In the approach the shape of the force spectrum can be parameterized and included as unknown in the estimation procedure. However, it is a very time-consuming approach.

The parameter estimation techniques mentioned until now have mainly been frequency domain techniques. However, the frequency domain techniques have several drawbacks. Firstly, rather long term records (data) are required to ensure a reliable frequency solution. Secondly, after producing e.g. so-called structural frequency response functions the parameters are not readily available. In fact, curve fitting algorithms must be subsequently used for this purpose. Thirdly,

when structures not lightly damped and with clustered modes are considered poor parameter estimates are usually obtained. Finally, it may be mentioned that spectral density functions estimated by Fourier transform techniques will always be biased. The bias in the frequency domain can be reduced but not removed, see e.g. Bendat et al. [36].

Through the quest for new identification techniques that could overcome the weakness of the Fourier techniques and become more suitable for digital processing, several time domain algorithms have been developed. Many of these algorithms seek to describe a dynamic system by means of an autoregressive-moving-average (ARMA) model. An ARMA(n, m) model of order n, m describing the response at the discrete time points y_t is given by

$$y_t = \sum_{i=1}^{n} \Phi_i y_{t-i} - \sum_{i=1}^{m} \mathcal{O}_i e_{t-i} + e_t \tag{3.23}$$

Φ_i is an Auto Regressive (AR) parameter, $\mathcal{O}_i$ is the Moving Average (MA) parameter and e_t is a time series of a white noise process. This model involves a difference equation in which the output of the system is expressed as linear combination of past output, as well as present and past input. This kind of model is particular well suited for identification and response calculation purposes since they provide efficient system representations. For many years the identification techniques based on ARMA models have attracted limited interest concerning structural engineering applications. A factor contributing to this situation is that ARMA models have been developed primarily by control engineers and applied mathematicians. Further, ARMA models have been primarily developed concerning systems for which limited a priori knowledge is available, whereas the identification of structural systems relies heavily on understanding of physical concepts.

In recent years the application of ARMA models to the description of structural systems has become more common, see e.g. Gersch et al. [37] Pandit et al. [38], Hac et al. [39], Natke [3] and Jensen [12]. The time domain identification techniques using ARMA representation have been compared with frequency domain techniques in e.g. Davies et al. [40]. In this and other papers it has been documented that these ARMA time domain modelling approaches are superior to Fourier approaches for the identification of structural systems. These foundings make identification techniques utilising ARMA algorithms interesting for modal parameter estimation.

It may be noticed that the ARMA models give a direct relation to the modal parameters while the Fourier methods give a non-parametric model which, followed by a curvefitting algorithm, give the estimates of the modal parameters. If an ARMA$(2n, 2n-1)$ model is used for a Gaussian white noise excited linear n-degrees-of-freedom system it can be shown that the covariance of the response due to the ARMA-model and that of the white noise excited structure will be identical, see e.g. Kozin et al. [8]. In other words, an ARMA model will provide an unbiased estimate of the autospectrum provided the assumptions hold. It is seen that parameter identification of civil engineering structures by using an

ARMA model assumes that the response data are caused by a white noise input to the structure. However, for wave or wind excited civil engineering structures, this assumption will normally hold, see section 3.2.

The modal parameters are obtained from the AR-parameters by minimizing an error function V_N expressing the variance of e_t

$$V_N = \frac{1}{N}\sum_{t=1}^{N} e_t^2 \tag{3.24}$$

It may be noticed that the white noise assumption must be checked when the AR and MA parameters and the residuals have been estimated. If the assumption does not hold it may indicate that the order of magnitude of the model is too low and therefore should be increased. The choice of order of magnitude of the model can be made as described in chapter 2, e.q. by using the Akaike's Information criterion.

When the AR parameters are estimated the dynamic parameters are found from the $2n$ roots, λ_i of the characteristic polynomial of the AR-parameters:

$$\lambda^{2n} - \Phi_1\lambda^{2n-1} - \cdots - \Phi_{2n-1}\lambda - \Phi_{2n} = 0 \tag{3.25}$$

In e.g. Pandit et al. [41] it is shown that the roots are related to the modal parameters through the $2n$ relations

$$\lambda_i = \exp(\mu_i \Delta t) \qquad i = 1, 2, .., 2n \tag{3.26}$$

where Δt is the sampling interval. μ_i has the following relation to the modal parameters for an undamped system

$$\mu_i = -\omega_i\zeta_i \pm i\omega_i\sqrt{1-\zeta_i^2} \qquad \zeta_i < 1.0 \tag{3.27}$$

By using the ARMA model all the information in the measured time series is used to estimate the AR-parameters. This implies that a large amount of data has to be handled in the system identification process implying that it can be time consuming to estimate the parameters. Especially, when the model order increases, caused of the non-linear optimization which has to be used to get the AR-parameters and the MA-parameters. However, Wolds [42] has shown that any ARMA model can be represented by an AR model if the model order is chosen sufficient high. This implies that the AR-parameters can be estimated directly by linear regression obtaining a least squares fit between the measured time series and the AR-model. The AR-parameters can also be obtained from estimates of the auto-correlation function. If (3.23) is multiplied at both sides by y_{t-k} and then take the expectation a difference equation for the auto-correlation function is obtained. This gives a set of linear equations in the AR-parameters often referred to as Yule-Walker equations. When the number of equations exceeds the number of parameters to be estimated, the system becomes overdetermined, and the

equations have no solution. For such situations, however, standard methods for determination of approximate solutions exist. One possibility is to solve the system of equations by least squares linear regression. In that case estimates of the AR-parameters are called overdetermined Yule-Walker estimates, see e.g. Söderström [43]. Using estimates of the auto-correlation function implies that the system identification process becomes less time-consuming since the auto-correlation function is estimated from a small amount of data compared to the original time series and because the estimates of the AR-parameters can be obtained by simple linear regression instead of optimization. In Brincker et al. [44] it is found, by investigation of a SDOF based on simulated data, that this technique of modal parameter estimation is almost as accurate as calibration of an ARMA model directly on the original time series, but faster, especially if the estimates of the auto-correlation function are obtained by the so-called Random Decrement Technique, see Brincker et al. [45].

3.5 Summary

This chapter has given an introduction to system identification of civil engineering structures. Further to the methods presented in this chapter many others exist, especially methods where both the response and input are assumed to be measured. However, in identification of civil engineering structures a method that is only based on the measured output has to be chosen, since it often can be difficult to measure the input signal.

From the chapter the following statements can be made:

- System identification in civil engineering is reduced to estimation of the parameters in an assumed model.
- It is common in parametric identification of civil engineering structures to use techniques based on measured output response and assuming the input to be white noise.
- Most identification methods used in civil engineering problems determines the modal parameters.
- Frequency domain methods have been dominating in the past while in the last decade applications of time domain techniques, such for instance ARMA models has increased.

Further, it can be concluded that

- The parameter estimates obtained by parametric identification of civil engineering structures are in general not presented in papers with their uncertainties. For validation or comparison it may be useful to know these.
- Design of system identification experiments in civil engineering seems to be nearly unconsidered.

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Chapter 4

Design of Optimal Input Signals for Parameter Estimation of Civil Engineering Structures

In the previous chapters, the basis of optimal experiment design for parameter estimation in models of dynamic systems has been presented. It is seen that the optimal choice of the experimental conditions can be performed based on the covariance matrix of the parameter estimates. Different ways to estimate the covariance matrix were discussed in chapter 2, while in chapter 3, the models normally used for parameter estimation of civil engineering structures were considered. Consequently, the foundation is created for considering the problem of optimal experiment design when a certain experimental condition, design variable, is chosen.

In this chapter, the optimal input design problem for parameter identification of civil engineering structures is considered. In spite of the fact, that ambient vibration measurements are common for testing of civil engineering structures, see chapter 3, it may be noticed that the choice of an excitation signal depends on the choice of system identification method. Therefore, if an experimenter wants to use a system identification method requiring an external excitation, the question arise: "How can an optimal external excitation signal be designed" ?. The input signals determine the operating point of the structure and the parts and modes of the structure which are excited during the system identification experiment. The choice of the input characteristics may vary with the application. Two different aspects are associated with the choice of input signal. One concerns the properties of the input signal such as its spectrum. The other concerns the shape of the input signal. In section 4.1 optimal input signal design theory is outlined. A state of the art will not be given, since it can be found elsewhere in the literature, see e.g. Mehra [1]. The aim of the section is to give a presentation of the basic methods available for optimal input signal design for parameter estimation of dynamic systems. Generally, some of the statements concerning optimal input signal design,

given in section 4.1, will also apply to experiment design, since the problem of optimal experiment design can be regarded as a generalisation of the problem of optimal input signal design. It may be noticed that it is mainly researchers working with electrical and control engineering who have developed the optimal input signal theory for parameter estimation of dynamic systems. In electrical engineering, the external input signal is fundamental for system identification experiments while in civil engineering system identification experiments by using the ambient excitation can be made. In order to show how the optimal input signal theory can be used for design of optimal input signals for parameter estimation of civil engineering structures, an example is given in section 4.2. In the example it is shown how an optimal stationary input signal can be designed for parameter estimation of a linear single degree-of-freedom mechanical system.

4.1. Design of Optimal Input Signals

In this section, the basic methods available for optimal input signal design for parameter estimation of dynamic systems will be presented. Most of the literature on input design for parameter estimation has been concentrated on the problem of obtaining accurate parameter estimates for parameters having some physical significance. This problem is related to design of input signals for parameter estimation in order to accurately predict the output of a system. This means that input signals should be chosen such that the transfer function of the system can be identified with the greatest accuracy. Further, parameter estimation for control system design, which is a subject in itself, is also related, see e.g. the state-of-the-art review of control system design in Soong [2]. This review concerns active control of civil engineering structures, a subject which has been given increasing consideration, since it was proposed in 1972 by Yao [3]. Active control implies an external excitation in order to obtain a vibration suppression of a structure which is dynamically loaded. Active control is an alternative to passive control which has been seriously considered in the recent 50 years. Passive control implies an arrangement which can reduce the response of the structure. The general subject concerning active control of dynamic systems has been studied longer than active control of civil engineering structures. For many decades, the basic concepts of active control have been the staple of electrical and control engineering and they have been applied successfully in a variety of disciplines, such as aerospace engineering and mechanical engineering. In section 4.1.1.1 the relation between the optimal input design for control and the optimal input design for parameter estimation of dynamic systems will be outlined. A more thorough outline can be found in e.g. Kalaba et al. [4] which is a book concerning the basic methods devoted to determination of optimal input for parameter estimation of dynamic systems and the optimal input for control.

A comprehensive survey of literature concerning the input signal design problem is given in e.g. Mehra [1], Zarrop [5], Kalaba [4] and Goodwin et al. [6]. Mehra

seems to be one the the most prolific researchers in the design of optimal input for parameter estimation of dynamic systems. According to the survey of optimal input signals design in Mehra [1], the first systematic attempt to solve the input signal design problem for time domain estimation is found in Levin [7] (1960).

In the existing literature, concerning the optimal input design, the researchers consider both time and frequency domain methods for a broad class of linear-nonlinear, continuous-discrete time and non-random single-input and single-output (SISO) systems and multi-input and multi-output systems (MIMO). With frequency domain design methods two different things can be indicated. Obviously, the choice of an input for spectral analysis will belong to these methods. However, also for time domain identification methods it is often useful to evaluate the accuracy of the obtained parameter estimates by means of frequency domain expressions and to characterise the optimal input signal in terms of its spectrum.

If the aim is to design an input signal for spectral analysis it is obvious that the autospectral density $S_{UU}(\omega)$ of the input signal $u(t)$ may not vanish for the frequencies $\omega_{min} \leq \omega \leq \omega_{max}$ if e.g. the so-called complex frequency response function $H(\imath\omega)$ given by (4.1) has to be determined

$$H(\imath\omega) = \frac{S_{UY}(\imath\omega)}{S_{UU}(\omega)} \tag{4.1}$$

$S_{UY}(\imath\omega)$ is the cross-spectral density between output $y(t)$ and the input $u(t)$. $\imath^2 = -1$. Notice that this does not necessarily mean that the true input spectrum has to have this properties. For the true discrete spectrum it is sufficient that it has positive values at a certain number of frequencies. From this discussion of input design it can be said that a minimal requirement to an input signal is that the input signal has to be sufficiently rich to excite all modes of interest during the experiments. This leads to the requirement of persistently exciting input signals. By definition a stationary input signal $u(t)$ is said to be persistently exciting of any order if, see e.g. Ljung [8]

$$S_{UU}(\omega) > 0 \qquad \forall : \omega \tag{4.2}$$

Further, it is shown in Ljung [8] that an informative experiment, defined in chapter 2, can be obtained for a system if the input signal is persistently exciting. Since the concept of persistent excitation is related to the informative experiments the minimal requirement properties of an input signal also establish the minimal requirement for parameter identifiability. A more thorough explanation of persistently exciting input signals of any order or of finite order can be found in e.g. Ljung [8] and Söderström et al. [9].

A study of the optimal input signal design for estimation of a frequency response function is given in Yuan et al. [10] and Ljung [8]. They consider the quality of the resulting transfer function estimate by using a quadratic norm of the difference between the transfer function obtained by prediction and by a chosen model, respectively. The problem is then to determine the optimal input signal so that the quadratic norm is minimized subjected to given constraints.

Above the minimal requirements for an input signal have been discussed. In the following, the maximal requirements for an input signal for parameter estimation will be discussed.

In the next sections, the following two input signal design problems will be considered

- Input Design for Time Domain Estimation, Time-Domain Approach.
- Input Design for Time Domain Estimation, Frequency-Domain Approach.

4.1.1 Input Design for Time Domain Estimation, Time-Domain Approach

In this section the general principle of the optimal input signal design for time domain estimation using a time domain approach is presented.

As mentioned in the introduction of this thesis the optimal input design is traditionally based on an efficient unbiased estimator implying that the covariance matrix $\overline{\overline{C}}_{\hat{\theta}_N}$ of the parameter estimate $\hat{\overline{\theta}}_N$ can be obtained, as mentioned in chapter 2, by the inverse of the Fisher information matrix $\overline{\overline{J}}$

$$\overline{\overline{C}}_{\hat{\theta}_N} \geq \overline{\overline{J}}^{-1} \tag{4.3}$$

This means that the existence of a parameter estimator giving at least asymptotically unbiased estimates with minimum variance has been assumed. An input signal design is then chosen so that a scalar measure of the inverse of the Fisher information matrix is made as small as possible. By doing so, a design is chosen to get as much information as possible about the parameter vector $\hat{\overline{\theta}}_N$ in a Fisherian sense. The design is based on the parameter covariance matrix, since the parameter estimates $\hat{\overline{\theta}}_N$ of the parameters to be estimated from the experiment depend on random processes, wherefore the accuracy of $\hat{\overline{\theta}}_N$ must be considered in a statistical sense.

An optimal input signal design, optimal in a Fisherian sense, can be formulated in the following way if a general time-invariant discrete SISO system is considered. A model of such a system can be written

$$y(t) = H(q, \overline{\theta})u(t) + G(q, \overline{\theta})e(t) \tag{4.4}$$

where q denotes a shift operator, i.e. $qu(t) = u(t+1)$. $H(q, \overline{\theta})$ and $G(q, \overline{\theta})$ are differentiable functions of the parameter vector $\overline{\theta}$. Generally $y(t)$ and $u(t)$ are realisations of stochastic output $\{Y(t)\}$ and input $\{U(t)\}$ processes, respectively. The variables $e(t)$ are normally assumed to be realisations of independent Gaussian distributed random variables $\{\mathcal{E}(t)\}$ with zero mean values and covariance $\lambda_{\mathcal{E}}$. $\{U(t)\}$ and $\{\mathcal{E}(t)\}$ are assumed to be independent which means that no feedback is granted, i.e. a so-called open-loop system is considered. Under certain circumstances, it will be advantageous to design an optimal input signal by considering

a close-loop system, see e.g. Söderström et al. [9]. In a close-loop system the feedback is considered.

By combining (4.3) and (4.4) the following is obtained by using the results from chapter 2

$$\overline{\overline{C}}_{\hat{\bar{\theta}}_N} = \overline{\overline{J}}^{-1} = \frac{\lambda_\varepsilon}{N}\left(E[\overline{\Psi}(t,\bar{\theta})\overline{\Psi}^T(t,\bar{\theta})]\right)^{-1} \tag{4.5}$$

where a realization $\overline{\psi}(t,\bar{\theta})$ of the stochastic process $\{\overline{\Psi}(t,\bar{\theta})\}$ is given by

$$\overline{\psi}(t,\bar{\theta}) = \frac{\partial \epsilon(t,\bar{\theta})}{\partial \bar{\theta}} = -G^{-1}(q,\bar{\theta})\left(\frac{\partial G(q,\bar{\theta})}{\partial \bar{\theta}}\epsilon(t,\bar{\theta}) + \frac{\partial H(q,\bar{\theta})}{\partial \bar{\theta}}u(t,\bar{\theta})\right) \tag{4.6}$$

It may be remembered that (4.5) is valid when the chosen model gives a correct description of the system to be identified for $\bar{\theta} = \bar{\theta}_0$. $\bar{\theta}_0$ is the "true" parameter vector and the processes are stationary. $\epsilon(t,\bar{\theta})$ is the prediction error given by

$$\epsilon(t,\bar{\theta}) = -G^{-1}(q,\bar{\theta})\Big(y(t) - H(q,\bar{\theta})u(t)\Big) \tag{4.7}$$

The optimal input signal design problem can now be stated

$$\begin{aligned} \min_{u(t)} \quad & \mathcal{A}\left(\overline{\overline{C}}_{\hat{\bar{\theta}}_N}(u(t))\right) \\ s.t. \quad & u^l(t) \le u(t) \le u^u(t) \\ & E[U^2(t)] \le E_u \\ & E[Y^2(t)] \le E_y \end{aligned} \tag{4.8}$$

where $\mathcal{A}(\cdot)$ in a scalar measure normally used in optimal input signal design, or more general, in experiment design, see section 1.1.2. The argument $u(t)$ is appended to stress that the result will depend on the design variable, the input signal. $E[\cdot]$ is the expectation operator. $u(t)$ represents a set of all possible informative input signals, i.e. persistently exciting input signals. For a realistic design the optimization problem is subjected to constraints. $u^l(t)$ and $u^u(t)$ are lower (l) and upper (u) constraints on the input signal amplitude, respectively. E_u and E_y are constraints on the input and output energy, respectively. The constraints are given in order to ensure that the structure is not forced into an operating region where the model becomes invalid. I.e. if a linear model is assumed the input should not excite non-linearities of the structure. However, the constraints on the amplitude of the input signal is a trade off between choosing a large amplitude in order to get a good signal-to-noise ratio at the output and limiting the range of the output in order that the structure remains within an approximately linear region. The signal may also be bounded so that the signal can be realized by the specified actuators. It has no sense to design an input signal which is not practicable.

The generally nonlinear optimization problem (4.8) can be solved at least in principle by standard optimization procedures. However, the need to search for an optimal solution is a computationally disadvantage aspect of the time-domain input design theory.

4.1.1.1 Design of Optimal Input signals Using Mehra's Method

In the foregoing section, an optimization problem was formulated in order to design an optimal input signal. In this section it will be explained how such an optimization problem, in certain circumstances, can be solved. In this section, a MIMO system will be considered.

Since the first systematic attempt to obtain optimal inputs, Levin [7], different methods have been proposed in order to solve the input signal design problem for time domain estimation. It is not the aim of this thesis to survey these methods, since this has already been done, see e.g. Mehra [1]. However, in order to explain how an optimal input signal can be obtained one method will be mentioned. For simplicity, the method will be explained in relation to estimation of one parameter in a linear time-invariant continuous time system. The method will be mentioned, since the principle of the method is repeated in many other methods proposed to solve the optimal input signal design problem. I.e. input design problems where e.g. estimation of more than one parameter, non-linear systems, discrete in time systems etc. are considered.

The idea of the method, see Mehra [12], is that the problem of the optimal input design is formulated as a linear-quadratic optimal control problem. Therefore, before Mehra's method is presented, the optimal control problem is considered in order to outline the relation between the optimal input design for control and the optimal input design for parameter estimation. A more thorough explanation can be found in e.g. Kalaba et al. [4] which is a book concerning the basic approaches devoted to determination of optimal inputs for control and the optimal inputs for parameter estimation, respectively.

The problem of optimal control is concerned with finding the optimal input control force, which changes the state of a system, in order to achieve a desired objective. In active control of civil engineering systems it means that the vibration of the structure has to be minimized using an external excitation. The level of the vibration and the control forces $\overline{u}(t)$ is formulated mathematically by a so-called performance index or loss function. A typical optimal control problem consists of finding the control $u(t), 0 \leq t \leq T_f$ that minimizes the quadratic performance index I

$$I = \frac{1}{2} \int_0^{T_f} \left[\overline{z}^T(t) \overline{\overline{Q}}(t) \overline{z}(t) + \overline{u}^T(t) \overline{\overline{R}}(t) \overline{u}(t) \right] dt \tag{4.9}$$

which is seen to be a problem in the calculus of variations. $\overline{z}(t)$ is a state vector. The superscript T indicates vector or matrix transpose, the time interval $[0, T_f]$ is defined to be longer than that of the external excitation. $\overline{\overline{Q}}(t)$ is a positive

semi-definite matrix and $\overline{\overline{R}}(t)$ is a positive definite matrix. These matrices are referred to as weighting matrices whose magnitudes are assigned according to the relative importance attached to the state variables and to the control forces in the minimisation procedure. The assignment of large values to the elements of $\overline{\overline{Q}}(t)$ indicates that response reduction is given priority over the control forces, and the opposite is true, when the elements of $\overline{\overline{R}}(t)$ are large in comparison with those of $\overline{\overline{Q}}(t)$, Hence, by varying the relative magnitudes of $\overline{\overline{Q}}(t)$ and $\overline{\overline{R}}(t)$ one can harmonise the controller to achieve a proper trade-off between control effectiveness and control energy consumption.

When only the response variables are assumed to be measured in a control problem, the control configuration is referred to as a close-loop control, since the response is continuously monitored and this information is used to make continuous corrections to the applied control forces. An open-loop control results when the control forces are regulated only by the measured excitation. In this case, where the information of both the response quantities and excitations is utilised for control design, the term open-close-loop control is used in the literature, i.e. generally the optimal control forces include two terms, one depending on the current state (close-loop term) and one depending on the external excitation and independent of the state (open-loop term). Close-loop control is widely used and applicable when the first term is dominating in relation to the last term.

In order to obtain the relation, the control law, between the measured state and the optimal control force for either a close-loop, open-loop or an open-close-loop the optimal control problem given by (4.9) has to be solved. This can be done by using different approaches as stated in the classical control theory. One way to solve the control problem is to formulate a so-called two-point boundary-value problem (TPBVP) which can be numerically solved by different numerical methods where there are advantages and disadvantages to each method. Analytical solutions for these TPBVP can only be obtained for simple problems.

In Merha [12] it is shown that the design of optimal input signals for linear system identification involves the solution of a TPBVP when it is formulated as a linear quadratic control problem.

In order to present Mehra's method a linear MIMO second-order system with a mass matrix $\overline{\overline{M}}$ is considered

$$\overline{\overline{M}}\ddot{\overline{y}}(t) + \overline{\overline{C}}\dot{\overline{y}}(t) + \overline{\overline{K}}\overline{y}(t) = \overline{u}(t) \tag{4.10}$$

where the solution to the second order differential equation is the response $\overline{y}(t)$. $\overline{\overline{C}}$ is the damping matrix and $\overline{\overline{K}}$ is the stiffness matrix.

Using a state-space representation (4.10) can be written

$$\dot{\overline{z}}(t) = \overline{\overline{F}}\overline{z}(t) + \overline{\overline{G}}\overline{u}(t) \tag{4.11}$$

where the state vector is given by

$$\overline{z}(t) = \begin{bmatrix} \overline{y}(t) \\ \dot{\overline{y}}(t) \end{bmatrix} \tag{4.12}$$

and

$$\overline{\overline{F}} = \begin{bmatrix} 0 & \overline{\overline{1}} \\ -\overline{\overline{M}}^{-1}\overline{\overline{K}} & -\overline{\overline{M}}^{-1}\overline{\overline{C}} \end{bmatrix} \tag{4.13}$$

$$\overline{\overline{G}} = \begin{bmatrix} 0 \\ \overline{\overline{M}}^{-1} \end{bmatrix} \tag{4.14}$$

where it is assumed that $\overline{\overline{G}}$ includes a parameter θ to be identified. It is assumed that the measurements $y(t)$ can be expressed as

$$\overline{y}(t) = \overline{\overline{S}}\overline{z}(t) + \overline{e}(t) \tag{4.15}$$

where $\overline{\overline{S}}$ is a selection matrix. $\overline{e}(t)$ describes the measurement noise taken as a zero mean stationary white noise process $\{\overline{\mathcal{E}}(t)\}$ with a variance $\lambda_{\mathcal{E}}$. Therefore

$$E[\overline{\mathcal{E}}^T(t_1), \overline{\mathcal{E}}(t_2)] = \overline{\overline{\Lambda}}\delta(t_1 - t_2) \tag{4.16}$$

where $E[\cdot]$ is an expectation operator. $\delta(t_1 - t_2)$ is the Dirac delta function and $\overline{\overline{\Lambda}}$ is a diagonal matrix with the diagonal elements $\lambda_{\mathcal{E}}$.

Now, the Fisher Information Matrix, which is a scalar, can be written

$$J = \int_0^{T_f} \left(\frac{\partial \overline{z}(t)}{\partial \theta}\right)^T \overline{\overline{S}}^T \overline{\overline{\Lambda}}^{-1} \overline{\overline{S}} \left(\frac{\partial \overline{z}(t)}{\partial \theta}\right) dt \tag{4.17}$$

Then the optimal input to be determined such that the Fisher information matrix is maximized can be obtained. $\frac{\partial \overline{z}(t)}{\partial \theta}$ is taken from

$$\frac{\partial \dot{\overline{z}}(t)}{\partial \theta} = \overline{\overline{F}}\frac{\partial \overline{z}(t)}{\partial \theta} + \frac{\partial \overline{\overline{G}}}{\partial \theta}\overline{u}(t) \tag{4.18}$$

The input energy is assumed to be constrained

$$\int_0^{T_f} \overline{u}^T(t)\overline{u}(t)dt = E_u \tag{4.19}$$

The maximization of J subjected to the constraint (4.19) is equivalent to the minimization of the performance index I

$$I = \min_{u(t)} \frac{1}{2} \int_0^{T_f} -\left(\frac{\partial \overline{z}(t)}{\partial \theta}\right)^T \overline{\overline{S}}^T \overline{\overline{\Lambda}}^{-1} \overline{\overline{S}} \left(\frac{\partial \overline{z}(t)}{\partial \theta}\right) + q_L \left(\overline{u}^T(t)\overline{u}(t) - \frac{E_u}{T_f}\right) dt \tag{4.20}$$

where q_L is a Lagrange multiplier, see e.g. Gill et al. [13], chosen to satisfy the constraint (4.19). This linear-quadratic problem can now be minimized by using different methods. In Merha [12] it is done by formulating a TPBVP using the

Pontryagin maximum principle, see e.g. Kalaba et al. [4]. Utilising Pontryagin's maximum principle the so-called Hamiltonian function Ha is written

$$Ha = \frac{1}{2}\left[-\left(\frac{\partial \overline{z}(t)}{\partial \theta}\right)^T \overline{\overline{S}}^T \overline{\overline{\Lambda}}^{-1} \overline{\overline{S}} \left(\frac{\partial \overline{z}(t)}{\partial \theta}\right)\right] + \overline{\lambda}_c^T(t)\left(\overline{\overline{F}}\frac{\partial \overline{z}(t)}{\partial \theta} + \frac{\partial \overline{\overline{G}}}{\partial \theta}u(t)\right) \quad (4.21)$$

The so-called costate vector $\overline{\lambda}_c(t)$ is the solution of the vector differential equation

$$\dot{\overline{\lambda}}_c(t) = -\left(\frac{\partial Ha}{\partial\left(\frac{\partial \overline{z}(t)}{\partial \theta}\right)}\right)^T \quad (4.22)$$

Inserting (4.21) in (4.22) implies

$$\dot{\overline{\lambda}}_c(t) = \overline{\overline{S}}^T \overline{\overline{\Lambda}}^{-1} \overline{\overline{S}} \frac{\partial \overline{z}(t)}{\partial \theta} - (\overline{\overline{F}})^T \overline{\lambda}_c(t) \quad (4.23)$$

Using the stationary condition the input signal that maximizes Ha is

$$u(t) = -\frac{1}{q_L}\left(\frac{\partial \overline{\overline{G}}}{\partial \theta}\right)^T \overline{\lambda}_c(t) \quad (4.24)$$

Substituting (4.24) into (4.18)

$$\frac{\dot{\overline{z}}(t)}{\partial \theta} = \overline{\overline{F}}\frac{\partial \overline{z}(t)}{\partial \theta} - \frac{1}{q_L}\left(\frac{\partial \overline{\overline{G}}}{\partial \theta}\right)\left(\frac{\partial \overline{\overline{G}}}{\partial \theta}\right)^T \overline{\lambda}_c(t) \quad (4.25)$$

Now (4.23) and (4.25) give the TPBVP which can be written

$$\begin{bmatrix} \frac{\partial \dot{\overline{z}}(t)}{\partial \theta} \\ \dot{\overline{\lambda}}_c(t) \end{bmatrix} = \begin{bmatrix} \overline{\overline{F}} & -\frac{1}{q_L}\left(\frac{\partial \overline{\overline{G}}}{\partial \theta}\right)\left(\frac{\partial \overline{\overline{G}}}{\partial \theta}\right)^T \\ \overline{\overline{S}}^T \overline{\overline{\Lambda}}^{-1} \overline{\overline{S}} & -\overline{\overline{F}}^T \end{bmatrix} \begin{bmatrix} \frac{\partial \overline{z}(t)}{\partial \theta} \\ \overline{\lambda}_c(t) \end{bmatrix} \quad (4.26)$$

with the boundary conditions

$$\frac{\partial \overline{z}(0)}{\partial \theta} = 0 \quad \overline{\lambda}_c(T_f) = 0 \quad (4.27)$$

Since the boundary conditions are homogeneous it is seen that the solutions to the problem exist for certain values of the multiplier q_L which correspond to the eigenvalues of the linear TPBVP. In order to obtain the optimal input (4.24) the eigenvalues q_L and $\overline{\lambda}_c(t)$ in (4.26) have to be determined. The eigenvalues and with that the optimal input can be determined in a number of ways. In Mehra [12] different numerical methods are discussed. One of these will be considered in order to show how a TPBVP can be numerically solved.

To obtain the optimal input the Riccati matrix and the transition matrix are defined in Merha [12] as described in the following.

The solution of the set of linear first order differential equations (4.26) can be expressed in terms of the transition matrix $\overline{\overline{\Phi}}(t, q_L)$ in the following way , see e.g. Kalaba et al. [4]

$$\begin{bmatrix} \frac{\partial \overline{z}(t)}{\partial \theta} \\ \overline{\lambda}_c(t) \end{bmatrix} = \overline{\overline{\Phi}}(t, q_L) \begin{bmatrix} \frac{\partial \overline{z}(0)}{\partial \theta} \\ \overline{\lambda}_c(0) \end{bmatrix} \tag{4.28}$$

where $\overline{\overline{\Phi}}(t, q_L)$ satisfies

$$\dot{\overline{\overline{\Phi}}}(t, q_L) = \overline{\overline{A}}\,\overline{\overline{\Phi}}(t, q_L) \tag{4.29}$$

$$\overline{\overline{\Phi}}(0, q_L) = \overline{\overline{I}}$$

$\overline{\overline{I}}$ is the identity matrix and $\overline{\overline{A}}$ is given by

$$\overline{\overline{A}} = \begin{bmatrix} \overline{\overline{F}} & -\frac{1}{q_L}\left(\frac{\partial \overline{\overline{G}}}{\partial \theta}\right)\left(\frac{\partial \overline{\overline{G}}}{\partial \theta}\right)^T \\ \overline{\overline{S}}^T \overline{\overline{\Lambda}}^{-1} \overline{\overline{S}} & -\overline{\overline{F}}^T \end{bmatrix} \tag{4.30}$$

If the partitioned transition matrix is written

$$\overline{\overline{\Phi}}(t, q_L) = \begin{bmatrix} \overline{\overline{\Phi}}_{11}(t, q_L) & \overline{\overline{\Phi}}_{12}(t, q_L) \\ \overline{\overline{\Phi}}_{21}(t, q_L) & \overline{\overline{\Phi}}_{22}(t, q_L) \end{bmatrix} \tag{4.31}$$

following equation is obtained by using the boundary conditions

$$\overline{\lambda}_c(T_f) = \overline{\overline{\Phi}}_{22}(T_f, q_L)\overline{\lambda}_c(0) = 0 \tag{4.32}$$

It is seen from (4.32) that the eigenvalues q_L are functions of the interval length T_f. In Mehra [12] it is shown that the performance index (4.20) will be minimized by the largest value of q_L, i.e. the largest eigenvalue corresponding to a given time interval T_f has to be determined. In order to estimate these values a numerical solution generally has to be used. In Mehra [12] it is shown that a so-called matrix Riccati equation can be used when the boundary conditions are homogeneous. The principle of this method for solution of the TPBVP is similar to the embedding method known from e.g. determination of the critical length of buckling of nonlinear columns, see e.g. Kalaba et al. [4]. This embedding method requires the integration of only one initial-value differential equation for a given choice of eigenvalue. The integration of the equation proceeds until the integral becomes excessively large indicating that the critical column length has been reached for a given eigenvalue.

In order to obtain an equation that can be integrated forward in time Merha [12] defines a matrix Riccati equation $\frac{\partial \overline{z}(t)}{\partial \theta}$ by

$$\frac{\partial \overline{z}(t)}{\partial \theta} = \overline{\overline{P}}(t)\overline{\lambda}_c(t) \tag{4.33}$$

An equation for $\overline{\overline{P}}(t)$ is obtained by differentiating both sides of (4.33) and substituting from equation (4.26) and rearranging yields

$$\dot{\overline{\overline{P}}}(t) = \overline{\overline{F}}\,\overline{\overline{P}}(t) + \overline{\overline{P}}(t)\overline{\overline{F}}^T - \overline{\overline{P}}(t)\overline{\overline{S}}^T\overline{\overline{\Lambda}}^{-1}\overline{\overline{S}}\,\overline{\overline{P}}(t) - \frac{1}{q_L}\left(\frac{\partial\overline{\overline{G}}}{\partial\theta}\right)\left(\frac{\partial\overline{\overline{G}}}{\partial\theta}\right)^T \tag{4.34}$$

$$\overline{\overline{P}}(0) = 0 \tag{4.35}$$

(4.34) is the well-known matrix Riccati equation which is used for solutions of many control input problems. Here it is used in order to determine the critical time interval T_c corresponding to the largest eigenvalue of the two-point boundary-value problem. It can be shown that when a critical time interval is found for a particular choice of eigenvalue then the eigenvalue is the largest eigenvalue corresponding to a time interval T_f equal to the critical time interval T_c.

Now the optimal input can be obtained by integrating the matrix Riccati equation forward in time for a particular choice of an eigenvalue q_L. When the elements of $\overline{\overline{P}}(t)$ become very large the critical time length $T_f = T_c$ has been reached. The initial costate vector $\overline{\lambda}_c(0)$ is obtained from equation (4.32) as an eigenvector. Thereby the boundary conditions $\overline{\lambda}_c(0)$ are satisfied. A unique value of the eigevector $\overline{\lambda}_c(0)$ is found by using the normalization conditions of the input energy constraint (4.19). Finally, the equation (4.26) is integrated forward in time using $\overline{\lambda}_c(t) = 0$ obtained above and the optimal input is obtained utilizing equation (4.24). Numerical errors are introduced in the solution because the critical time length T_c and the eigenvector $\overline{\lambda}_c(0)$ cannot be determined exactly. The accuracy of the solution can be improved by integrating $\overline{\overline{P}}^{-1}(t)$ and then the critical time length to a particular eigenvalue is determined by a zero crossing.

If a particular time interval T_f is desired then the matrix Riccati equation must be integrated several times with different values of the eigenvalues q_L in order to determine the value of q_L corresponding to the desired T_f.

Above, it is explained how Mehra's method can be used to design an optimal input for estimation of one parameter in a linear time-invariant continuous time MIMO system with homogeneous boundary conditions. If the boundary conditions are non-homogeneous the Ricatti method is not available to solve the TPBVP. Mehra's method is also usable if more than one parameter have to be identified also if they are included in $\overline{\overline{F}}$. More than one parameter implies that a scalar measure has to be used, since $\overline{\overline{J}}$ is a matrix in the multi-parameter case. By using a trace measure the optimal input design can be formulated as a TPBVP which can be solved by one of the numerical methods available for solution of such problems as described in e.g. Mehra [12] or Kalaba et al. [4]. It may be noticed that different numerical methods, in addition to the matrix Ricatti method, for solution of the input design problem are discussed in Mehra [12]. If D-optimal design is used, in the multivariable case, the input signal design problem has to be solved otherwise than explained above. Mehra [12] proposes different algorithms for design of bounded energy input signals and bounded amplitude input signals, respectively.

It may also be noticed that on-line recursive input signal design algorithms for parameter estimation are developed, see e.g. Goodwin et al. [6]. However, such algorithms do not give optimal input signals in the same sense, Fisherian sense, as the methods described above. They are optimal in the one-step-ahead sense. Recursive algorithms for optimal input signal design for parameter estimation are related to the well-known algorithms, in control engineering, for active control (adaptive control) purposes, see e.g. Goodwin et al. [6]

From the above it is seen that design of an optimal input signal by using a time domain method can be cumbersome and problem dependent. Only considered has been an introduction to the problem in order to present the basic design principles.

In the following section it will be seen that certain simplifications occur for the input design signal problem in the limit as the measurement time gets long and if the class of input signals is restricted to those having a spectral representation.

4.1.2 Input Design for Time Domain Estimation, Frequency-Domain Approach

The expression for the covariance of the parameter estimates (4.5)-(4.7) can be expressed by frequency functions by using Parseval's theorem. By using the theorem stating that the energy of a signal in the time domain must equal the energy computed in the frequency domain the following expression for the Fisher information matrix is obtained, see e.g. Ljung [8]

$$\begin{aligned}\overline{\overline{J}} =& \frac{N}{\lambda\varepsilon}\int_{-\pi}^{\pi} |G(e^{\imath\omega},\bar{\theta}_0)|^{-2}\left(\frac{\partial H(e^{\imath\omega},\bar{\theta}_0)}{\partial\bar{\theta}_0}\right)\left(\frac{\partial H(e^{-\imath\omega},\bar{\theta}_0)}{\partial\bar{\theta}_0}\right)^T S_{UU}(\omega)d\omega + \\ & \frac{N}{\lambda\varepsilon}\int_{-\pi}^{\pi} |G(e^{\imath\omega},\bar{\theta}_0)|^{-2}\left(\frac{\partial G(e^{\imath\omega},\bar{\theta}_0)}{\partial\bar{\theta}_0}\right)\left(\frac{\partial G(e^{-\imath\omega},\bar{\theta}_0)}{\partial\bar{\theta}_0}\right)^T d\omega \qquad (4.36)\end{aligned}$$

where $e^{\imath\omega}$ is the exponential function taken to the argument $\imath\omega$ where $\imath^2 = -1$. $S_{UU}(\omega)$ is the discrete input spectrum defined by

$$S_{UU}(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} R_{UU}(k)e^{-ik\omega} \qquad -\pi < \omega < \pi \qquad (4.37)$$

which implies that the discrete autocovariance is given by

$$R_{UU}(k) = \int_{-\pi}^{\pi} S_{UU}(\omega)e^{ik\omega}d\omega \qquad (4.38)$$

It may be noticed that some authors use $\frac{1}{2\pi}$ in eq. (4.38) and not in eq. (4.37). This implies that in some references the expression for the parameter information matrix (4.36) includes a factor $\frac{1}{2\pi}$. It is easily seen from (4.36) that the right hand side is real.

Using that the noise autospectrum is given by

$$S_{\varepsilon\varepsilon}(\omega) = \frac{\lambda_\varepsilon}{2\pi}|G(e^{\imath\omega})|^2 \tag{4.39}$$

it is seen that the parameter covariance matrix in the frequency domain can now be written

$$\overline{\overline{C}}_{\hat{\bar{\theta}}_N} \approx \frac{S_{\varepsilon\varepsilon}(\omega)}{N}\left(\int_{-\pi}^{\pi}\left(\frac{\partial H(e^{\imath\omega},\bar{\theta}_0)}{\partial\bar{\theta}_0}\right)\left(\frac{\partial H(e^{-\imath\omega},\bar{\theta}_0)}{\partial\bar{\theta}_0}\right)^T S_{UU}(\omega)d\omega + \int_{-\pi}^{\pi}\left(\frac{\partial G(e^{\imath\omega},\bar{\theta}_0)}{\partial\bar{\theta}_0}\right)\left(\frac{\partial G(e^{-\imath\omega},\bar{\theta}_0)}{\partial\bar{\theta}_0}\right)^T d\omega\right)^{-1} \tag{4.40}$$

From (4.40) it may directly be seen how the input signal properties, i.e. the input spectrum $S_{UU}(\omega)$ affect the accuracy of the obtained parameter estimates. It is seen that to achieve a small covariance matrix, the input energy should be applied to frequencies where the sensitivity of the response functions is large. It is also seen that the parameter accuracy depends only on the input spectrum, i.e. different input signals having the same spectra will thus yield the same information matrix. This is a useful result, since it implies that one can first determine the best input spectrum and then select a realization of that spectrum taking practical aspects of the signal generation and input limits into account.

The optimal input in the frequency domain can now be stated as

$$\begin{aligned} \min_{S_{UU}(\omega)} \quad & \mathcal{A}\left(\overline{\overline{C}}_{\hat{\bar{\theta}}_N}(S_{UU}(\omega))\right) \\ s.t. \quad & E[U^2(t)] \leq E_u \\ & E[Y^2(t)] \leq E_y \end{aligned} \tag{4.41}$$

where $\mathcal{A}(\cdot)$ is a scalar measure of the covariance matrix of the parameter estimates.

(4.41) can usually only be solved by numerical minimization. To facilitate the numerical mimimization it will be useful to have a finite-dimensional parameterization of the input spectrum. Typical parametrizations are in terms of finite sums of sinusioids. However, realization of the optimal input spectrum can also be given by other types of signal than sinusoids. The literature concerning design of input signals for system identification deals with a number of different types of signal beyond sum of sinusoids, e.g. white or coloured noise sequences, step inputs, impulse inputs and pseudorandom binary sequences. A pseudo-random sequence is a signal that shifts between two levels in a certain fashion. Examples and a further discussion of these different kinds of signal can be found in e.g. Söderström et al. [9].

If (4.41) is subjected to following input constraint

$$E_u = \int_{-\pi}^{\pi} S_{UU}(\omega)d\omega \tag{4.42}$$

some simplifications in the design of input signal for parameter estimation of a dynamic system are obtained. The input constraint implies that any information matrix can be obtained by applying an input which is simply a sum of a finite number $(n_\theta(n_\theta+1)/2+1)$ of sinusoids of various amplitudes, see Goodwin et al. [6]. Further, it is shown in Zarrop [5] that the information matrix is non-singular if the input signal exhibits more than $n_\theta/2$ sinusoids. n_θ is the number of parameters, i.e. an input signal design for a SISO involves at most $(n_\theta(n_\theta+1)+2)$ search variables (amplitudes and frequencies), in general far less than the search variables (number of points in a input time series) involved in a general time domain design using the time domain approach. These results indicate that the input design problem is more easily handled in the frequency-domain than in the time-domain.

The problem to determine the optimal input signal consisting of a finite number of sinusiods for parameter estimation of a SISO has been considered by many researchers see e.g. Goodwin et al. [6], Jaherivan [14], Mehra [15] and Zarrop [5]. In Zarrop [5] it is shown that D-optimum designs could be achieved for both input or output power constraint cases by minimum $n_\theta/2$ single sinusoids or maximum $(n_\theta+1)/2$ if a SISO system is considered. He also gives algorithms to estimate the optimal frequencies an their weights. Such algorithms can also be found in e.g. Goodwin et al. [16]. By using a computer study Javaherian [14] has also obtained that a D-optimum design can be achieved for maximum $(n_\theta+1)/2$ input frequencies. In Mehra [15] a method is developed for designing optimal input signals for linear systems using a frequency method. The method yields optimal frequencies and amplitudes for the optimal input signal when a D-optimal design is chosen.

Above it is seen how optimal input signals can be designed either by using a time domain method or a frequency domain method. Again it may be noticed that the methods are mainly developed for electrical engineering purposes where an external input is fundamental for system identification. However, the methods have been used in parameter estimation experiments for aircraft parameter estimation, see e.g. Plaetschke et al. [16]. Use of the design methods in civil engineering seems to be sparse. Further, if an external excitation is wanted good solutions can be obtained based on intuitive reasoning. However, if an optimal input signal is required it is seen in this chapter that appropriate design methods are available. In order to show how to use the design method for a civil engineering problem a simple example is given in the following.

4.2 Example 4.1: Optimal Design of A Stationary Input Signal

In order to demonstrate the above-mentioned methods the following example will deal with the estimation of an optimal frequency of a sinusiodal input signal $u(t)$ for parameter estimation of a linear single degree of freedom mechanical vibrating system with a mass constant m

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = \frac{u(t)}{m} \tag{4.43}$$

where the solution to the second order differential equation is the response $y(t)$. ω_n is the undamped natural angular frequency and ζ is the damping ratio.
The parameters to be estimated are given by parameter vector $\overline{\theta} = [\omega_n, \zeta]$.
It is assumed that the displacement measurements $y(t)$ can be expressed as

$$y(t) = y(t|\overline{\theta}) + e(t) \tag{4.44}$$

where $e(t)$ describes the measurement noise taken as a zero mean stationary white noise process $\{\mathcal{E}(t)\}$ with a variance $\lambda_\mathcal{E}$. Therefore

$$E[\mathcal{E}(t_1), \mathcal{E}(t_2)] = \lambda_\mathcal{E}\delta(t_1 - t_2) \tag{4.45}$$

where $\delta(t_1 - t_2)$ is the Dirac delta function.
For the system in (4.44) it is seen that, see e.g. Thomsen [17]

$$\frac{S_{YY}(\omega)}{S_{UU}(\omega)} = \frac{1}{m^2((\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega^2\omega_n^2)} = |H(\imath\omega)|^2 \tag{4.46}$$

where $H(i\omega)$ is the so-called complex frequency function given by

$$H(i\omega) = \frac{1}{m((\omega_n^2 - \omega^2) + 2\zeta\omega\omega_n i)} \tag{4.47}$$

$S_{YY}(\omega)$ and $S_{UU}(\omega)$ are the autospectra for the response and the excitation, respectively.
Using the results stated in Zarrop [5], the optimal excitation for parameter estimation of $\overline{\theta}$ can be obtained by using an excitation $u(t)$ given by

$$u(t) = \sin\omega_0 t \tag{4.48}$$

which is a single sinusoidal wave with frequency ω_0. The autospectrum for this excitation is given by

$$S_{UU}(\omega) = \frac{1}{4}(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)) \tag{4.49}$$

The real parts of the derivatives in (4.40) are given by

$$\left(\frac{\partial H(\imath\omega,\bar{\theta})}{\partial\omega_n}\right)\left(\frac{H(-\imath\omega,\bar{\theta})}{\partial\omega_n}\right)=\frac{4m^2\omega_n^2+4\zeta^2\omega^2m^2}{(m^2((\omega_n^2-\omega^2)^2+4\zeta^2\omega^2\omega_n^2))^2} \tag{4.50}$$

$$\left(\frac{\partial H(\imath\omega,\bar{\theta})}{\partial\zeta}\right)\left(\frac{\partial H(-\imath\omega,\bar{\theta})}{\partial\zeta}\right)=\frac{4\omega_n^2\omega^2m^2}{(m^2((\omega_n^2-\omega^2)^2+4\zeta^2\omega^2\omega_n^2))^2} \tag{4.51}$$

$$\left(\frac{\partial H(\imath\omega,\bar{\theta})}{\partial\omega_n}\right)\left(\frac{\partial H(-\imath\omega,\bar{\theta})}{\partial\zeta}\right)=\frac{4\zeta\omega^2m^2\omega_n}{(m^2((\omega_n^2-\omega^2)^2+4\zeta^2\omega^2\omega_n^2))^2} \tag{4.52}$$

In this example it is assumed that a continuous time system is considered. This means that the integral in (4.40) over the frequency range $[0;\pi]$ is replaced by integration over $[0;\infty]$. Further, $e^{\imath\omega}$ is replaced by $\imath\omega$.

The optimal frequency ω_0^{opt} is obtained by substituting (4.47),(4.59),(4.50),(4.51) and (4.52) in (4.40). If a D-optimum design is selected the determinant of the covariance matrix except a constant (*const*) is given by (4.53)

$$\det[\overline{\overline{C}}_{\hat{\theta}_N}]=const\left(\frac{16\omega_n^4m^4\omega_0^2}{(m^2((\omega_n^2-\omega_0^2)^2+4\zeta^2\omega_0^2\omega_n^2))^4}\right)^{-1} \tag{4.53}$$

Minimum of (4.53) is obtained for the optimal frequency ω_0^{opt} given by

$$\omega_0^{opt}=\omega_n\sqrt{\frac{6}{14}}\sqrt{(1-2\zeta^2)+\sqrt{(1-2\zeta^2)^2+\frac{7}{9}}} \tag{4.54}$$

It is seen from the expression that the optimal frequency depends on the parameters to be estimated. This means, as mentioned before, that this optimal input signal design necessitates the knowledge of some a priori estimates of the unknown parameters of the parameters to be identified. However, it will seen later that this is a typical result in experiment design for parameter estimation.

The variation of the normalized optimal input frequency to variation of the damping rate is shown in figure 4.1. by the line named "Determinant Criterion". It is seen as expected that the optimal frequency varies inversely with the damping ratio. Further it is seen that the optimal frequency is the undamped natural angular frequency if the damping ratio $\zeta \to 0$.

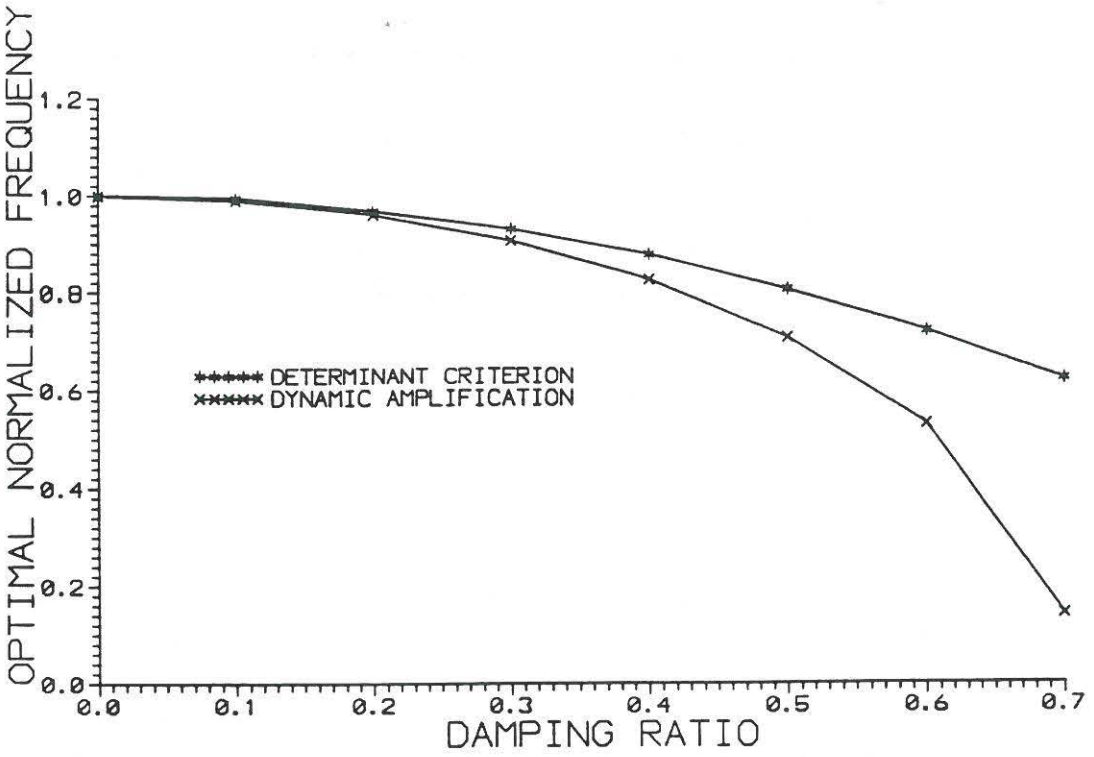

Figure 4.1: Optimal normalized input frequency to variation of damping.

In figure 4.1 the line named "Dynamic Amplification" shows the normalized input frequency given by

$$\omega_0^{opt} = \omega_n \sqrt{1 - 2\zeta^2} \tag{4.55}$$

This is the frequency giving maximum of dynamic amplification, i.e. the frequency corresponding to a maximum of the sensitivity $\frac{\partial |H(i\omega,\bar{\theta})|}{\partial \zeta}$

The difference between the optimal frequency based on the information matrix and the optimal frequency based on maximum dynamic amplification is seen to be insignificant in the range of damping ratio [0.01;0.05] for a civil engineering structure. However, it is seen that for increasing damping ratio the difference becomes more significant. The result shows that it suffices to base design of a system identification experiment on intuitive reasoning, e.g. to use an input giving maximum of dynamic amplification. However, if optimal information return is desirable it is seen that design methods are available as outlined above. It may be noticed that in this example the optimal number, (one), is determined from the results in Zarrop [5] where a relation between the number of parameters to be identified and the optimal number of excitation frequencies are given for a SISO system. Such results do not seem to be available for a MIMO system. So if design optimal input signals for parameter estimation of MIMO systems is the object, a computer study has to be made, see the one Jaherivan [14] has made for a SISO system. Further, for an optimal excitation of a MINO structural system the problem of optimal location of actuators also has to be considered. This problem corresponds to the problem of optimal location of measurement points which will to be considered in chapter 6.

It may be noticed that in this chapter optimal input signal design is discussed. However, how to choose between experiments based on ambient excitation or external excitation has not been considered. In principle, the choice between external or ambient excitation is a cost-benefit problem where the cost of using external excitation has to be taken into account when the system identification experiment is designed. Thus in order to solve such a problem it is necessary to establish a method to estimate the value of the extra information obtained by using external excitation instead of ambient excitation. The problem of taking costs into account is discussed in chapters 7 and 8.

4.3 Summary

In this chapter, the problem to determine an optimal input signal for parameter estimation of dynamic systems has been discussed. The main observations can be stated as follows:

- Ambient testing in civil engineering is commonly used for identification of civil engineering structures while an input signal design does not have the same fundamental importance as in electrical engineering. However, if an optimal input signal is wanted it is seen that appropriate design methods are available.
- Input signal design is traditionally based on an assumed asymptotically efficient unbiased estimator implying that the covariance matrix of the parameters to be identified can be established from the Fisher information matrix. An optimal input is then obtained by minimizing a scalar measure of the covariance matrix giving optimal input signals in a Fisherian sense.
- Parameter identifiability is secured by persistent excitation.
- Design of input signals for time domain identification can be made by time domain methods or frequency methods. Design is more easily handled in the frequency domain than in the time domain when stationary solutions are usable. It may be noticed that many of the results given in the literature for design of optimal input signals only hold true for special cases, such as efficient estimators and long measuring time.
- Designed input signals should be capable of being implemented and respect constraints under which the model is valid.
- The input signal design depends on the parameters to be identified and therefore, good a priori knowledge about the parameters is fundamental.
- The choice between external or ambient excitation is a cost-benefit problem where the cost of using external excitation has to be taken into account when the system identification experiment is designed. In order to solve such a problem it is necessary to have a method to estimate the value of the extra information obtained by using external excitation instead of ambient excitation.

4.4 References

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Chapter 5

On the Choice of Optimal Sampling Interval and Experiment Length

The final acquisition of data, produced by the system to be identified, is to be made on digital computers. The analysis of the recorded analog time signal is mainly limited to the time during the measuring in order to control the quality of the measurements. By using such digital equipment a discretisation of the measured continuous signal is the result. Therefore, it is unavoidable that sampling as such leads to information losses and it is important to select the sampling intervals, so that these losses are insignificant. The intuition says that the higher sampling rate the better a discrete representation of the continuously measured signal will be obtained. However, in this chapter it will be shown that this is not true in general.

The aim of the chapter is to investigate how a sampling interval can be selected in order to minimize the information losses about the parameters to be identified in a parameter estimation experiment. Optimal experiments, which are optimal in a Fisherian sense, will be considered. This means that the same optimization problem as in chapter 4 has to be solved. The optimization variable is the sampling interval Δt instead of the input signal $u(t)$. However, in this chapter an example will also be shown where a joint design is established with both the sampling interval and the input spectrum as design variables. It may be noticed that only uniform sampling will be considered in this chapter. See e.g. Goodwin et al. [1] for a discussion of non-uniform sampling. Before discussing in detail the discretisation of continuously measured signals it is worthwhile to mention the role of anti aliasing filters. In order to avoid the so-called alias phenomenon (folding the spectral density) it is important that any continuous signal which is sampled, should first have been filtered to remove frequencies which are too high to be represented in the sampled signal. In section 5.1 this problem (aliasing), arising in signal processing when a discretisation is performed of the measured continuous signal will be considered. Next, in section 5.2 and 5.3, respectively, two examples are given in order to investigate the choice of sampling interval. The

first example considers the joint determination of input spectrum, pre-sampling filter and sampling interval. The second example is concerned with the problem of the choice of sampling interval and experiment length if the modal parameters by an ARMA-model for a white noise loaded structure modelled as a single degree-of-freedom linear mechanical system is to be estimated. Section 5.4 will discuss choice of sampling interval and experiment length in parameter estimation methods where a model not directly fitted to the original time series is used. In stead of the estimates of the parameter are obtained direct from estimates of e.g. spectra or covariance functions.

5.1 On the Discretisation Problem

The random vibration of structures is normally measured by an equipment, a sensor, that produces an analog voltage signal proportional to acceleration. In order to prepare this raw data for final system identification analysis different phases in data acquisition and processing may be done. These different phases are discussed in detail in e.g. the Shock & Vibration Handbook [2], Bendat et al. [3] and Bendat et al. [4]. Here the problem of data preparation will be considered, more precisely the step in data preparation which is the conversion of the measured analog data to a digital format (digitisation).

Before considering the choice of discretisation, sampling interval (Δt) it can be helpful to review the well-known Shannon sampling theorem, see e.g. Bendat et al. [3].

Sampling Theorem: *Suppose that a continuous signal is sampled at equally spaced intervals Δt then the only frequencies which may be reconstructed from the sampled signal are those up to half of the sample frequency, the so-called Nyquist frequency.*

If the sampling frequency is denoted $\omega_s = 2\pi/\Delta t$ the sampling theorem implies that the Nyquist frequency ω_N, or folding frequency has to satisfy (5.1)

$$\omega_N < \frac{\omega_s}{2} = \frac{\pi}{\Delta t} \tag{5.1}$$

In order to get a unique representation of the frequency content in the measured signal it is necessary to satisfy (5.1). If a component of the measured signal is beyond the Nyquist frequency this information will be lost. In fact the situation is worse than this because not only is the information lost but aliasing (folding) will occur. This means that the part of the measured signal spectrum corresponding to frequencies higher than ω_N will be interpreted as contributions from lower frequencies when the measured data are analysed. The aliasing problem is illustrated in figure 5.1.

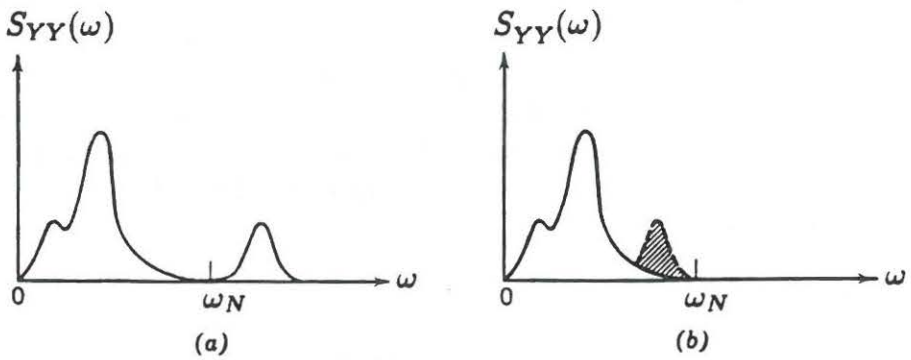

Figure 5.1: Aliased autospectrum due to folding. (a) True spectrum. (b) Aliased spectrum.

In spite of the fact that the information about the frequencies higher than the Nyquist frequency is lost by sampling it is then important not to make matters worse by let the folding effect distort the interesting part of the measured spectrum below the Nyquist frequency. This is achieved by using a so-called analog antialiasing filter. Such a filter removes that information in the original analog data that might exist at frequencies above the Nyquist frequency prior to the analog-to-digital conversion.

Ideally, such a so-called analog low-pass presampling filter $F(\imath\omega)$ should then have a gain satisfying

$$|F(\imath\omega)| = 1 \qquad \omega \leq |\omega_N| \tag{5.2}$$

$$|F(\imath\omega)| = 0 \qquad \omega > |\omega_N| \tag{5.3}$$

which the continuous analog signal is passing before sampling. By using such a filter the following relation between the spectrum of the filtered signal $S^F_{YY}(\omega)$ and the spectrum of the continuous signal $S_{YY}(\omega)$ would be obtained

$$S^F_{YY}(\omega) = |F(\imath\omega)|^2 S_{YY}(\omega) \tag{5.4}$$

This implies that the filtered spectrum is given by

$$S^F_{YY}(\omega) = S_{YY}(\omega) \qquad -\omega_N \leq \omega \leq \omega_N \tag{5.5}$$

and does not include any aliasing effect. An antialiasing filter should always be applied before sampling if it is suspected that the measured signal has non negligible energy above the Nyquist frequency.

It may be noticed that in practice the filter $F(\imath\omega)$ only can be realized approximately. According to Ljung [5] it is not possible for an analog anti aliasing filter to

have filter characteristics with a nearly flat amplitude response up to the cut-off frequency followed by a sharp drop without some phase distortion in the frequency range below the cut-off frequency of the filter. The more sharp the filter is with respect to the amplitude the more distorted the phase response will be. A ideal filter does not have a phase distortion. However, the filter should be designed so that high-frequency content of the measured signal above the Nyquist frequency is well damped and the low-frequency content (the interesting part) is not very much affected. A typical situation is that the signal consists of a useful part and a disturbance part and that the spectrum of the disturbances is more broadbanded than that of the signal. Then the sampling interval is usually chosen so that of the spectrum of the useful part is below the cut off frequency of the filter. The anti aliasing filter then essentially cuts away the high-frequency disturbance contributions. It is easy to show by using the expression for the Fisher information matrix in the frequency domain that an experiment with uniform sampled data cannot be better than the corresponding continuous experiment. This result is of course quite expected. However, if an ideal antialiasing filter is used it can be shown that there is no loss of information by performing the discretisation, see e.g. Goodwin et al. [1] or Payne et al. [6]. It is assumed that the input spectrum is band limited and no energy over the cut-off frequency of the anti aliasing filter and that a linear problem is considered.

Specific analog anti aliasing filters are proposed in the literature, e.g. so-called Butterworth and Chebyshev antialiasing filters. In figure 5.2 the characteristics of an analog Butterworth filter is shown for increasing model order and it is seen from the above-mentioned problem that for increasing model phase distortion is becoming more clear.

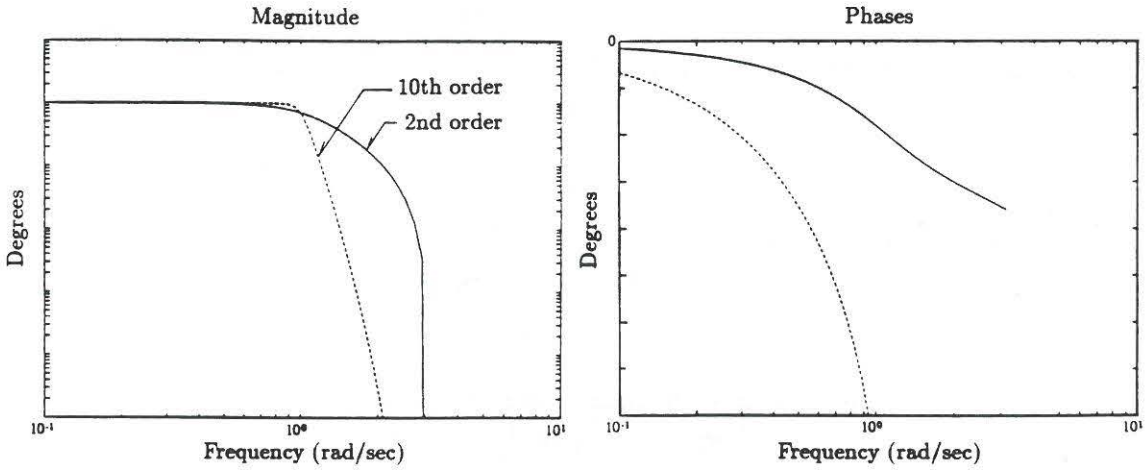

Figure 5.2: Characteristics of a Butterworth analog antialiasing filter shown for different model orders.

For a more thorough discussion of anti aliasing filter, see e.g. Bendat et al. [2], Shock & Vibration Handbook [2] and MATLAB [7].

Until now there have been concentrated on the aliasing problem. In the following the choice of sampling interval will be discussed. It will be seen that an optimal sampling interval can be found under certain circumstances. The discussion of the choice of sampling interval is given in the following two examples.

5.2 Example 5.1: Joint Optimal Determination of the Input Spectrum, Presampling Filter and Sampling Interval

A design method for joint optimal determination of the input spectrum, presampling filter and sampling interval for the case of constraint input and fixed total number of sample points N is considered. The design method is outlined in Goodwin et al. [1] and Payne et al. [6]. This simple example is given in order to show how to make joint design of identification experiments. Further, the result of the example shows that the choice of sampling interval depends on the choice of input spectrum.

The method will be illustrated by the same model as used in example 4.1. In example 4.1 the optimal frequency of a sinusoidal input $u(t)$ for parameter estimation of a linear single degree of freedom mechanical vibrating system is considered. The one degree-of-freedom system is modelled by

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = \frac{u(t)}{m} \tag{5.6}$$

where the solution to the second order differential equation is the the response $y(t)$. The parameters to be identified are the damped natural angular frequency ω_n and the damping ratio ζ. m is a mass constant.

Since the sampling frequency is ω_s the total experiment time T_f is given by

$$T_f = N\Delta t = \frac{N2\pi}{\omega_s} \tag{5.7}$$

Then the Fisher information matrix $\overline{\overline{J}}$ can be written

$$\overline{\overline{J}} = \frac{N2\pi}{\omega_s}\overline{\overline{J}}^s_{av} \tag{5.8}$$

$\overline{\overline{J}}^s_{av}$ is the average information matrix per unit time corresponding to an input spectrum having highest frequency ω_h and samples collected at frequency

$$\omega_s > 2\omega_h \tag{5.9}$$

If an ideal antialiasing filter is used it was mentioned above that by including it prior to sampling ensures that the subsequent sampling will not deteriorate the information matrix. Hence (5.8) can be written

$$\overline{\overline{J}} = \frac{N2\pi}{\omega_s}\overline{\overline{J}}_{av} \tag{5.10}$$

where $\overline{\overline{J}}_{av}$ is the average information matrix corresponding to continuous observations. This information matrix can be obtained from the information matrix established in example 4.1 by dividing by the experiment length.

Since the frequency of the sinusoidal excitation signal is ω_0 (5.10) becomes

$$\overline{\overline{J}} = \frac{N\pi}{\omega_0}\overline{\overline{J}}_{av} \tag{5.11}$$

since sampling is made just above the Nyquist frequency, twice the input frequency.

By using the expression for $\overline{\overline{J}}_{av}$ obtained in example 4.1 it is, except a constant *const*, obtained that the determinant of the inverse of the information matrix becomes

$$\det[\overline{\overline{J}}^{-1}] = const\left(\frac{16\omega_n^4 m^4 \omega_0^2}{(m^2((\omega_n^2-\omega_0^2)^2+4\zeta^2\omega_0^2\omega_n^2))^4}\frac{1}{\omega_0^2}\right)^{-1} \tag{5.12}$$

Minimum of (5.12) is obtained for the optimal frequency ω_0^{opt} given by

$$\omega_0^{opt} = \omega_n\sqrt{1-2\zeta^2} \tag{5.13}$$

It is seen that this optimal excitation frequency corresponds to the frequency corresponding to maximum of dynamic amplification in example 4.1. I.e. that the result of the joint design problem implies that the system should be excited by ω_0^{opt} with optimal sampling frequency $\omega_s = 2\omega_0^{opt}$

It is also seen that the optimal input frequency in the discrete problem is suppressed compared with the optimal frequency in the continuous problem. Further, it is seen that the problem of choice of sampling rate is closely related to the problem of choosing the input signal. A change of input signal may very likely cause a change of the optimal sampling rate.

5.3 Example 5.2: Optimal Choice of Sampling Interval for Identifying a Civil Engineering Structure Loaded by White Noise

From the foregoing chapters it is seen that a common form of test for identifying civil engineering structures is ambient testing where a white noise assumption for the ambient excitation is commonly used. Then from chapter 3 it is seen that ARMA models for identifying civil engineering structures are usable. This example will consider the choice of sampling interval for an ambient test for identifying a civil engineering structure by using an ARMA-model. It will be seen that an analytical solution can be obtained for the relation between sampling interval and the Fisher information matrix. The alternative would have been a simulation study which can be time-consuming.

Consider the following single degree-of-freedom mechanical vibrating system where the stochastic response process$\{Y(t)\}$ is the solution to the second order differential equation

$$\ddot{Y}(t) + 2\zeta(2\pi f)\dot{Y}(t) + (2\pi f)^2 Y(t) = Q(t) \tag{5.14}$$

f is the eigenfrequency implying that the natural undamped frequency $\omega_n = 2\pi f$. ζ is the damping ratio and $\{Q(t)\}$ is stationary zero mean Gaussian white noise.

From chapter 3 it is seen that a proper discrete model for the second order SDOF continuous system excited by white noise is an ARMA(2,1) model. This discrete model is given by

$$y_t = \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + e_t - \mathcal{O}_1 e_{t-1} \tag{5.15}$$

where the discrete time, the sampling interval, is Δt. Φ_1, Φ_2 is the Auto Regressive (AR) parameters, $\mathcal{O}_1$ is the Moving Average (MA) parameter and $e(t)$ is a time series of independent Gaussian distributed numbers. The ARMA parameters are given by, see e.g. Pandit et al. [8]

$$\Phi_1 = 2e^{-\zeta\omega_n\Delta t}\cos(\omega_d\Delta t) \tag{5.16}$$

$$\Phi_2 = -e^{-2\zeta\omega_n\Delta t} \tag{5.17}$$

$$\mathcal{O}_1 = -P \pm \sqrt{P^2 - 1} \tag{5.18}$$

where

$$P = \frac{\omega_n \sinh(2\zeta\omega_n\Delta t) - \zeta\omega_n \sin(2\omega_d\Delta t)}{2\zeta\omega_n \sin(\omega_d\Delta t)\cosh(\zeta\omega_n\Delta t) - 2\omega_d \sinh(\zeta\omega_n\Delta t)\cos(\omega_d\Delta t)} \tag{5.19}$$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency.

The problem is now whether there is an optimal sampling rate or not if the parameters $\overline{\theta} = [f, \zeta]^T$ should be estimated from an experiment . This problem will be considered in the following.

From measurements of the response process it is possible to get unbiased estimates of the AR-parameters Φ_1 and Φ_2, see e.g. Pandit et al. [8] , where estimates of the variances of the estimated parameters can be estimated by the Cramer-Rao lower bound. This implies that the covariance matrix of parameter estimates can be obtained by the inverse of the Fisher information matrix $\overline{\overline{J}}$ which can be written

$$\overline{\overline{J}} = \frac{\lambda_\varepsilon}{N}\left(E[\Psi(t,\overline{\Phi})\Psi^T(t,\overline{\Phi})]\right) \tag{5.20}$$

where a realization of the stochastic process $\{\Psi(t,\overline{\Phi})\}$is given by

$$\psi(t,\overline{\Phi}) = \frac{\partial\epsilon(t,\overline{\Phi})}{\partial\overline{\Phi}} \tag{5.21}$$

It is assumed that the variance of the noise process $\{\mathcal{E}(t)\}$ is λ_ε. N is the number of samples. $\overline{\Phi}$ is a vector including the AR-parameters

The prediction error $\epsilon(t, \overline{\Phi})$ is given by

$$\epsilon(t, \overline{\Phi}) = \frac{(1 - \Phi_1 q^{-1} - \Phi_2 q^{-2})y(t)}{1 - \mathcal{O}_1 q^{-1}} \tag{5.22}$$

where q is a shift operator. From (5.21) and (5.22) the following is obtained

$$\frac{\partial \epsilon(t, \overline{\Phi})}{\partial \Phi_1} = \frac{-q^{-1}}{1 - \mathcal{O}_1 q^{-1}} y(t) \tag{5.23}$$

$$\frac{\partial \epsilon(t, \overline{\Phi})}{\partial \Phi_2} = \frac{-q^{-2}}{1 - \mathcal{O}_1 q^{-1}} y(t) \tag{5.24}$$

Since (5.15) is assumed to give a true description of the system in (5.14)

$$\frac{\partial \epsilon(t, \overline{\Phi})}{\partial \Phi_1} = \frac{-q^{-1}}{1 - \Phi_1 q^{-1} - \Phi_2 q^{-2}} e(t) \tag{5.25}$$

$$\frac{\partial \epsilon(t, \overline{\Phi})}{\partial \Phi_2} = \frac{-q^{-2}}{1 - \Phi_1 q^{-1} - \Phi_2 q^{-2}} e(t) \tag{5.26}$$

is obtained.

It is now possible to calculate $\overline{\overline{J}}^{-1}$ from (5.20), (5.25) and (5.26) in the following way. The Fisher information matrix is

$$\overline{\overline{J}} = \frac{\lambda_\varepsilon}{N} \begin{bmatrix} E[\frac{\partial \epsilon(t,\overline{\Phi})}{\partial \Phi_1}]^2 & E[\frac{\partial \epsilon(t,\overline{\Phi})}{\partial \Phi_1} \frac{\partial \epsilon(t,\overline{\Phi})}{\partial \Phi_2}] \\ E[\frac{\partial \epsilon(t,\overline{\Phi})}{\partial \Phi_1} \frac{\partial \epsilon(t,\overline{\Phi})}{\partial \Phi_2}] & E[\frac{\partial \epsilon(t,\overline{\Phi})}{\partial \Phi_2}]^2 \end{bmatrix} \tag{5.27}$$

The elements in the information matrix can be written

$$E\left(\frac{\partial \epsilon(t, \overline{\Phi})}{\partial \Phi_1}\right)^2 = \frac{\lambda_\varepsilon}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i\omega}}{1 - \Phi_1 e^{-i\omega} - \Phi_2 e^{-2i\omega}} \frac{e^{i\omega}}{1 - \Phi_1 e^{i\omega} - \Phi_2 e^{2i\omega}} d\omega \tag{5.28}$$

$$E\left(\frac{\partial \epsilon(t, \overline{\Phi})}{\partial \Phi_2}\right)^2 = \frac{\lambda_\varepsilon}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-2i\omega}}{1 - \Phi_1 e^{-i\omega} - \Phi_2 e^{-2i\omega}} \frac{e^{2i\omega}}{1 - \Phi_1 e^{i\omega} - \Phi_2 e^{2i\omega}} d\omega \tag{5.29}$$

$$E\left(\frac{\partial \epsilon(t, \overline{\Phi})}{\partial \Phi_1} \frac{\partial \epsilon(t, \overline{\Phi})}{\partial \Phi_2}\right) = \frac{\lambda_\varepsilon}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i\omega}}{1 - \Phi_1 e^{-i\omega} - \Phi_2 e^{-2i\omega}} \frac{e^{2i\omega}}{1 - \Phi_1 e^{i\omega} - \Phi_2 e^{2i\omega}} d\omega \tag{5.30}$$

It is seen from (5.27), (5.28), (5.29) and (5.30) that the information matrix does not depend on the variance of the noise process.

Using that the complex number z is given by

$$z = e^{i\omega} \tag{5.31}$$

and

$$dz = ie^{i\omega} d\omega = iz d\omega \tag{5.32}$$

(5.28) can be written

$$E\left(\frac{\partial\epsilon(t,\overline{\Phi})}{\partial\Phi_1}\right)^2 = \frac{1}{2\pi i}\oint \frac{z^{-1}}{1-\Phi_1 z^{-1}-\Phi_2 z^{-2}}\frac{z}{1-\Phi_1 z^1-\Phi_2 z^2}\frac{dz}{z} \tag{5.33}$$

$\oint$ is a complex integration around the unit circle $|z| = 1$, counterclockwise. The complex integration around the unit circle can now be evaluated by using residue calculus, see e.g. Kreyzig [9]

$$\oint f(z)dz = 2\pi i \sum_{j=1}^{k} Res f(z_j) \tag{5.34}$$

where $z_j = 1, 2, .., k$ are the singular points inside the unit circle. The singular points in (5.34) are

$$z_1 = \frac{\Phi_1}{2} + \frac{1}{2}\sqrt{\Phi_1^2 + 4\Phi_2} \tag{5.35}$$

$$z_2 = \frac{\Phi_1}{2} - \frac{1}{2}\sqrt{\Phi_1^2 + 4\Phi_2} \tag{5.36}$$

$$z_3 = \frac{\Phi_1}{2\Phi_2} + \frac{1}{2\Phi_2}\sqrt{\Phi_1^2 + 4\Phi_2} \tag{5.37}$$

$$z_4 = \frac{\Phi_1}{2\Phi_2} - \frac{1}{2\Phi_2}\sqrt{\Phi_1^2 + 4\Phi_2} \tag{5.38}$$

From above it is seen that

$$\Phi_1^2 + 4\Phi_2 \leq 0 \tag{5.39}$$

given that f and Δt are both positive and real as long as ζ stays positive and real. This implies that the complex roots are complex conjugate pairs and

$$|z_1| = |z_2| = e^{-\zeta\omega\Delta t} < 1; \Delta t > 0 \tag{5.40}$$

$$|z_3| = |z_4| = e^{\zeta\omega\Delta t} > 1; \Delta t > 0 \tag{5.41}$$

This means that the roots z_1 and z_2 are singular points within the unit circle while z_3 and z_4 are singular points outside the unit circle, i.e. it is only the singular points z_1 and z_2 which shall be taken into account when the complex integral in (5.33) is calculated.

The calculation of the integral can be made by using see e.g. Kreyzig [9]

$$Res f(z_j) = Res\frac{p(z_j)}{q(z_j)} = \frac{p(z_j)}{q'(z_j)} \tag{5.42}$$

where $'$ denotes a derivative of $q(z)$ with respect to z.

Above it is explained how one element in the information matrix can be calculated. The other elements can be calculated in the same way.

When the elements of the information matrix are calculated the parameter covariance matrix $\overline{\overline{C}}_{\hat{\theta}_N}$ of estimates of the parameter vector $\hat{\theta}_N$ can be expressed in the following way

$$\overline{\overline{C}}_{\hat{\theta}_N} = \overline{\overline{A}}\,\overline{\overline{J}}^{-1}\overline{\overline{A}}^T \tag{5.43}$$

where the transformation matrix $\overline{\overline{A}}$ is given by

$$\overline{\overline{A}} = \begin{bmatrix} \frac{\partial f}{\partial \Phi_1} & \frac{\partial f}{\partial \Phi_2} \\ \frac{\partial \zeta}{\partial \Phi_1} & \frac{\partial \zeta}{\partial \Phi_2} \end{bmatrix} \tag{5.44}$$

Since the connection between the AR-parameters and the parameters $\overline{\theta} = [f, \zeta]^T$ is non-linear the transformation matrix $\overline{\overline{A}}$ cannot be directly obtained. Instead of following relation is used

$$\overline{\overline{B}}\,\overline{\overline{A}} = \overline{\overline{I}} \tag{5.45}$$

where $\overline{\overline{I}}$ is the identity matrix. Then

$$\overline{\overline{A}} = \overline{\overline{B}}^{-1} \tag{5.46}$$

The matrix $\overline{\overline{B}}$ is given by

$$\overline{\overline{B}} = \begin{bmatrix} \frac{\partial \Phi_1}{\partial f} & \frac{\partial \Phi_1}{\partial \zeta} \\ \frac{\partial \Phi_2}{\partial f} & \frac{\partial \Phi_2}{\partial \zeta} \end{bmatrix} \tag{5.47}$$

The above estimation of $\overline{\overline{A}}$ will only be accurate if the function is sufficiently smooth since it corresponds to a linear approximation of the function describing the inverse transformation from AR- parameters to the parameters $\overline{\theta}$.

The covariance matrix of $\overline{\theta}$ is now expressed as a function of the sampling interval Δt. This analytical connection makes it relatively easy to consider the problem whether there is an optimal sampling rate or not.

5.3.1 Results

A simulation study is performed in order to investigate the applicability of the analytical solution.

The most accurate way to perform simulations of an SDOF system formulated in continuous time is to transform the system model into discrete time space which can be done by using the ARMA(2,1) model described above. By using this model the response is simulated for an SDOF system. The parameters are estimated by an ARMA(2,1) model and the expected values $\mu_{\hat{\theta}_N}$ of the parameter estimates $\hat{\theta}_N$ and the covariance matrix $\overline{\overline{C}}_{\hat{\theta}_N}$ is estimated by the sample averages

$$\mu_{\hat{\theta}_N} = \frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} \hat{\theta}_N^{(i)} \tag{5.48}$$

$$\overline{\overline{C}}_{\hat{\theta}_N} = \frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} (\hat{\theta}_N^{(i)} - \mu_{\hat{\theta}_N})(\hat{\theta}_N^{(i)} - \mu_{\hat{\theta}_N})^T \tag{5.49}$$

where N_{sim} is the number of simulations. The simulation study is performed using the MATLAB software package on a VAX 8700 computer. A description of the the MATLAB software packages can be found in [7].

In table 5.1 the parameter estimation uncertainty based on simulations (SIM) of the response is shown together with values obtained by the analytical (ANA) solution for an SDOF system. δ_f and δ_ζ are the coefficient of variation of the eigenfrequency and the damping, respectively. $\rho_{f,\zeta}$ is the correlation coefficient between the the eigenfrequency and the damping.

The simulation study is performed with following data:

f=1 Hz.

ζ=0.005, 0.02 and 0.05

$\Delta t = 0.3$

N_{sim}=100

N=8000

	ANA	SIM	ANA	SIM	ANA	SIM
ζ	0.005	0.005	0.02	0.02	0.05	0.05
δ_f	0.00058	0.00063	0.00125	0.00130	0.00193	0.00203
δ_ζ	0.1163	0.1253	0.0598	0.0599	0.0401	0.0416
$\rho_{f,\zeta}$	0.0032	0.1665	0.0142	0.0619	0.0424	0.0322

Table 5.1: Parameter estimation uncertainty based on simulations (SIM) of the response is shown together with values obtained by the analytical (ANA) solution for an SDOF system.

Comparing the analytical and the simulated results for the coefficient of variation it is seen that the analytical results predict rather well what is to be expected in the practical simulation. It is seen that they deviate less than 10 per cent. On the other hand, it is seen that the prediction of the analytical correlation coefficient estimates is fairly uncertain. Thus, it can be concluded that the theoretical values obtained for the parameter uncertainty give a good indication of the parameter uncertainty obtained from practical simulations. Further, it is seen that the coefficient of variation of the eigenfrequency is proportional with the damping ratio and that the coefficient of variation of the damping ratio is inverse proportional with the damping ratio.

In the following the analytical solution will be used to investigate the problem whether there is an optimal sampling interval or not. The SDOF system is modelled by the eigenfrequency and damping ratio mentioned above. The number of data points is N=8000.

In figure 5.3 the coefficient of variation of the eigenfrequency and the damping ratio is shown for different sampling intervals Δt. The sampling interval is varied

from $0 \rightarrow 0.5$ corresponding to the Nyquist frequency being equal to the resonance frequency of the system. Considering the variation coefficients as functions of the sampling interval it is seen that they have minima for certain values of Δt. It is also seen that functions are flat near the minima. This causes difficulties in the precise choice of the optimal sampling interval on the one hand, but it also means that some imperfections in the optimally chosen sampling interval result in relatively small increase in error. However, it is seen that the coefficients of variation increase rapidly when the sampling interval Δt increases from the optimal value giving the minima. Thus, it is far worse to use a too large Δt than a too small one when the number of data points N is constant. This result corresponds to the result given in Åström [10] where a first order system is analysed. The result that an optimal sampling interval exists when N is fixed has also been obtained in Jensen et al. [11] by a simulation study of a second order system excited by white noise.

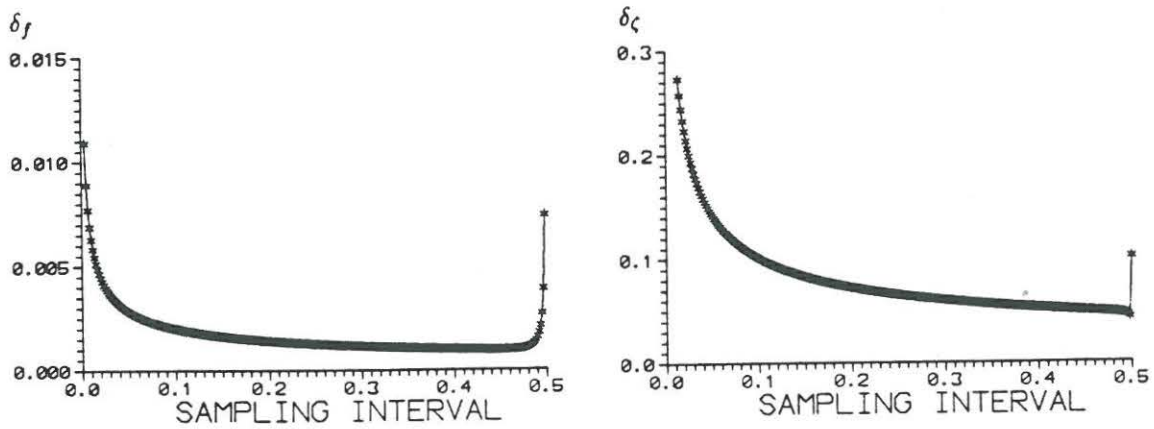

Figure 5.3: The coefficients of variation δ_f and δ_ζ shown as functions of Δt. ($N = 8000$)

In figure 5.4 the coefficient of variation of the eigenfrequency and the damping ratio is shown for different sampling intervals Δt, where N is kept constant and the total experiment length $T = N\Delta t$ is varied. Here T=1000 sec. For this case it is seen that there is no optimal sampling interval. The coefficients of variation are monotonically increasing when the sampling interval increases. The figure shows that the smaller the sampling interval the smaller the uncertainty of the parameter estimates.

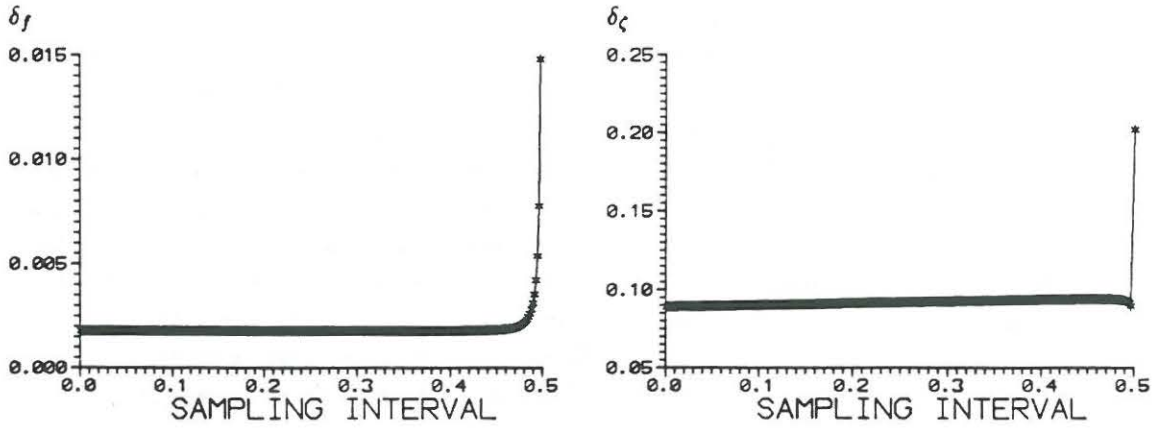

Figure 5.4: The coefficients of variation δ_f and δ_ζ shown as functions of Δt. (T=1000 sec.)

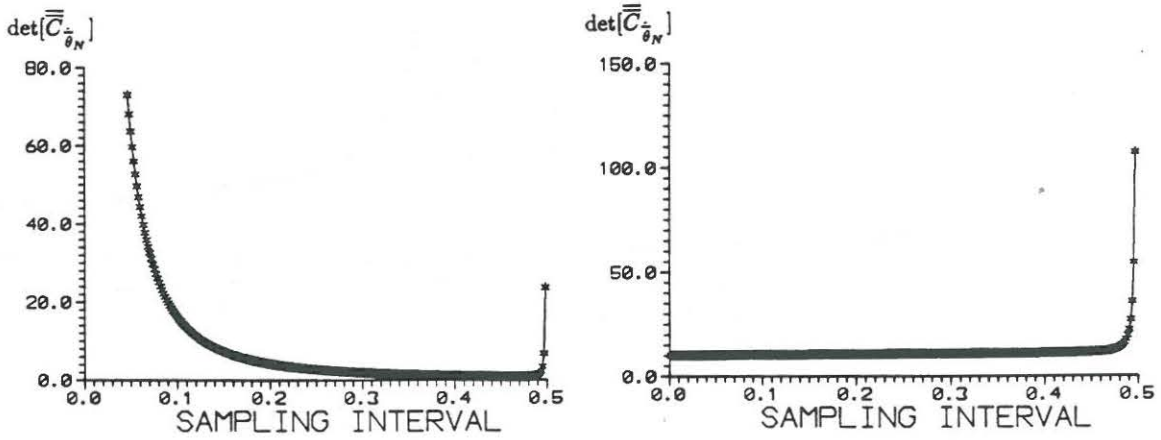

Figure 5.5: The determinant of the parameter covariance matrix shown as a function of Δt.

In figure 5.5 the determinant of the parameter covariance matrix is shown for different sampling intervals Δt. To the left in figure 5.5 N is constant and to the right T is constant. It is seen that an optimal sampling interval exists for N constant and not for T constant.

Table 5.2 shows the optimal sampling intervals Δt^{opt} corresponding to the figures shown above. Further the optimal sampling interval obtained by using a trace (tr) scalar measure of the parameter covariance is also shown. The third and fourth column show the optimal sampling interval corresponding to a minimum of the coefficient of variation of the eigenfrequency and damping ratio, respectively.

	det	tr	δ_f	δ_ζ
Δt^{opt}	0.4621 sec.	0.4561 sec.	0.4449 sec.	0.4997 sec.

Table 5.2: Optimal sampling interval for different design criteria

Table 5.2 shows that the choice of optimal sampling interval depends on the design criteria. However, it is seen that the optimal choice of the sampling interval only deviates a little for the different criteria.

In order to investigate the sensitivity of the optimal sampling interval to a variation of the damping ratio and the eigenfrequency f for a fixed number of samples N the dimensionless quantity $\Delta t^{opt} f$ is calculated for different values of ζ, see figure 5.6. The optimal sampling intervals are determined by the determinant of the parameter covariance matrix.

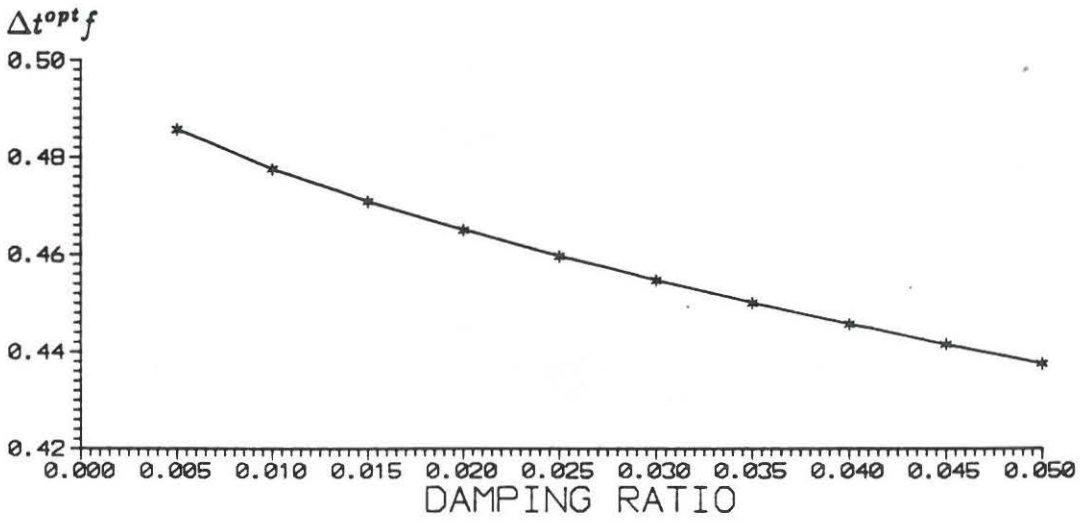

Figure 5.6: The dimensionless quantity $\Delta t^{opt} f$ shown as function of Δt.

From the results used in figure 5.6 the dimensionless quantity $\Delta t^{opt} f$ is fitted by a linear expression giving

$$\Delta t^{opt} = (0.4876 - 1.045\zeta) f^{-1} \tag{5.50}$$

It is seen that the optimal sampling interval is proportional to ζ^{-1} and f^{-1}.

Until now the optimal choice of the sampling interval has been investigated. However, the choice of experiment length has not been considered. In practice, the experiment length is often limited due to stationarity requirements or purely to practical considerations. The restriction on the number of data is frequently met in practice due to the cost of data acquisition or computer storage restrictions. Sampling one measured signal at e.g. 100 Hz, it will take just 20-21 minutes to fill up 1 Mb. However, it may be noticed that another convenient type of data storage system is the analog or digital recorder. Such an equipment has the advantages of being able to store large quantities of data and to reproduce them on electrical form. If the data storage permits and if cheap data acquisition is dealt with, sampling at a high sampling rate can be made. This data may then be filtered and desampled to the desired frequency before performing the final data analysis. Thus the following is obtained

- sampling interval for data acquisition,
- sampling interval for final parameter estimation.

However, it is seen from figure 5.7 that when the experiment length, (number of sample points) has reached a given magnitude only limited improvement can be obtained by increasing N. One way to determine the optimal choice of N could be a cost-benefit analysis. In chapter 8 this problem will be considered.

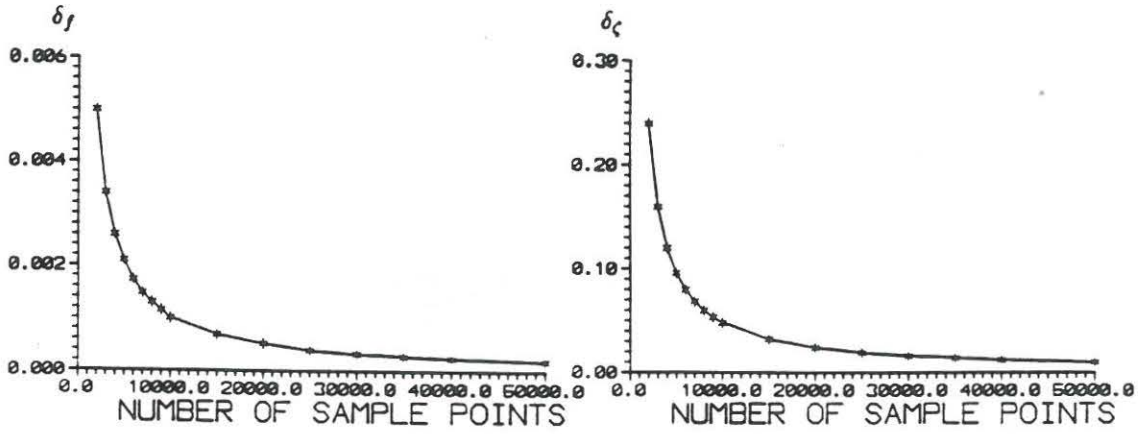

Figure 5.7: The influence of number of sample points on the coefficients of variation of the eigenfrequency and damping ratio.

Above it is shown how the sampling interval can be determined in different situations. It may be noticed that when analytical calculations like those in this example are too laborious to carry out or other model structures are necessary, simulations can be used to estimate the best sampling rate. However, such simulations may be time-consuming.

5.4 Design of Experiments for Parameter Estimation Based on Basic Estimates

In the foregoing sections, the choice of optimal sampling interval and experiment length for parameter estimation has been discussed when direct identification methods are used, i.e. when system identification methods are directly used on the measured time series. However, indirect system identification methods can also be used in order to obtain estimates of the parameter to be identified. I.e. first e.g. spectra or covariance functions are estimated and then the parameter is obtained from these basic estimates. In chapter 3 it is described that parameter estimation methods for civil engineering structures is available by such indirect methods. E.g. the spectral moment methods based on estimates of the response spectra or the Yule- Walker method based on auto-correlation estimates of the measured response. The reason why such methods can be preferable for system identification is, as mentioned in chapter 3, that they are computernationally superiority to system identification methods where models are directly fitted to the measured time series. However, since such basic estimates are obtained from measured realisations of random processes they have to be interpreted as estimates of random variables. The accuracy of such estimates based on sampled values can be described by a bias and a random error

$$bias\ error = E[\hat{\varsigma}] - \varsigma \tag{5.51}$$

$$random\ error = \sqrt{E[\hat{\varsigma}^2] - E^2[\hat{\varsigma}]} \tag{5.52}$$

where $\hat{\varsigma}$ is an estimator for ς. However, in order to determine estimates of the system parameters from the basic estimates it is important to have basic estimates which have a small bias error or if possible an unbiased one. According to their definition the auto-correlation function estimates are unbiased, see e.g. Bendat et al. [3]. On the other hand, the autospectral density function estimated by Fourier transform methods has a biased error. Therefore, as mentioned in chapter 3, it can be advantageous to use system identification methods based on auto-correlation estimates instead of spectral function estimates if system identification has to be based on basic estimates.

In order to investigate the optimal number of experiment length, sampling interval etc. prior to an experiment where system identification methods based on basic estimates has to be used, a simulation study can be performed. However such a simulation study can be time-consuming. Instead the possibility of establishing an analytical solution in the same way as in section 5.3 could be used. Such a solution gives a relation between the uncertainty of the basic estimates and the covariance matrix of the parameter estimates to be determined from the basic estimates. In e.g. Söderström et al. [12] it is shown how such a solution can be obtained for a simple problem.

The uncertainty of the basic estimates is investigated in e.g Bendat et al. [3] where error expressions are given for the basic estimates, i.e. closed solutions as

function of the experiment design variables such as experiment length, number of samples etc. are given. Further, it can be mentioned that the variance and bias contribution to the error in estimation of transfer functions of linear systems is investigated in Gevers et al. [13] and Wahlberg et al. [14], respectively.

Instead of investigating the error of the parameter estimates as functions of the experiment design variables for design of an optimal experiment only the error of the basic estimates could be estimated. Thus, the design of the experiment for parameter estimation is based on the error of the basic estimates. Such an investigation could be performed by considering the expressions for error functions given in Bendat et al. [3]. This will be considered in the following example.

5.5 Example 5.3: On the Choice of Experimental Length and Resolution Bandwidth

In this example it is assumed that the parameter estimation experiment is going to be based on autospectral estimates.

The system to be considered is modelled as a single degree-of-freedom system excited by white noise. Then the displacement response of the system will have an autospectral density $S_{YY}(\omega)$ given by, see e.g. Thomson [15],

$$S_{YY}(\omega) = \frac{S_0}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega^2\omega_n^2} \tag{5.53}$$

where S_0 is the autospectral density of the white noise process

The normalised bias error δ_b and random error δ_r of (5.53), respectively, can approximately be written, see Bendat et al. [3],

$$\delta_b \approx \frac{B_e^2 \frac{d^2 S_{YY}(\omega)}{d\omega^2}}{24 S_{YY}(\omega)} \tag{5.54}$$

$$\delta_r \approx \frac{1}{\sqrt{B_e T_f}} \tag{5.55}$$

Where B_e and T_f are the resolution bandwidth and the experiment length, respectively.

It may be noticed that the expression for the bias error will overestimate the error for sharp peaks in the autospectral density function. Nevertheless, due to a second order Taylor expansion, (5.54) constitutes a useful first-order approximation that correctly describes important qualitative results. It is seen that the bias error increases as $\frac{dS_{YY}(\omega)}{d\omega}$ increases for a given B_e or as B_e increases for a given $\frac{dS_{YY}(\omega)}{d\omega}$.

The expression for the normalised random error is based on the assumption that the measured data are obtained from a stationary Gaussian random process. The

expression shows that the normalised random error is a function of the experiment length T_f and the resolution bandwidth B_e only. This implies that the experimental length needed to provide a specified normalised random error in an autospectral density function estimate can be determined if the resolution bandwidth is known and the bias error is negligible. In practice, the random error of the autospectral density function is reduced by computing an ensemble of estimates from n_d different subrecords each of length T_d and averaging the results to obtain a final "smooth" estimate for the autospectrum.

The total normalised mean square error δ_t of the autospectral density function estimate can be written

$$\delta_t \approx \frac{1}{B_e T_f} + \frac{B_e^4}{576}\left[\frac{\frac{d^2 S_{YY}(\omega)}{d\omega^2}}{S_{YY}(\omega)}\right]^2 \tag{5.56}$$

Two important features of this error expression should be noted. First, there are conflicting requirements for the resolution bandwidth B_e, namely, a small value of B_e is needed to suppress the bias portion of the error while a large value of B_e is desired to reduce the random portion of the error. Secondly, the random portion of the error includes only B_e and not the total data bandwidth. Hence the random portion of error is a function primarily of the analysis parameters rather than unknown data parameters. This enhances the practical value of (5.56) in experimental design.

In figure 5.8 the total normalised error of the autospectral density function to variations of the experimental length T_f and the resolution bandwidth B_e is shown for the single degree-of-freedom system. The system is assumed to have an eigenfrequency $f = 1$ Hz and damping ratio $\zeta = 0.02$.

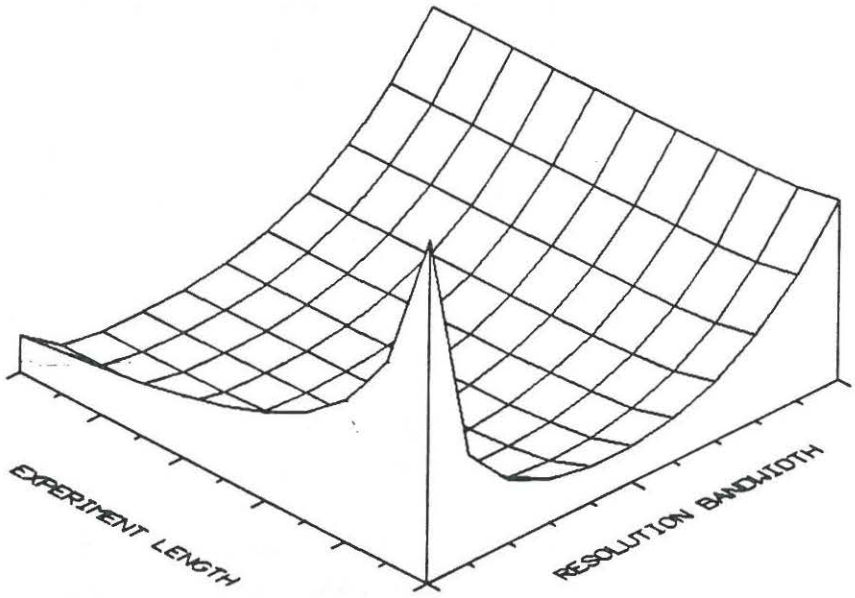

Figure 5.8: The total normalised error of the autospectral density function to variations of the experimental length T_f and the resolution bandwidth B_e.

Figure 5.8 shows that the experimental length has to be large and the resolution bandwidth has to be small if a lightly damped structure should be identified. However, as mentioned above, the experiment is often limited due to stationarity requirements or purely to practical considerations. Then, if the experimental length is "small" an optimal resolution bandwidth has to be determined. Here, the expressions for uncertainty of the autospectral density function has been considered. Such expressions also exist for other kind of basic estimates such as auto-correlation functions, cross-correlation functions etc., see Bendat et al. [3].

5.6 Summary

- Suppose that a continuous signal is sampled at equally spaced intervals Δt. Then the only frequencies which may be reconstructed from the sampled signal are those up to half of the sample frequency.
- Prefiltering of data is often necessary to avoid aliasing (folding of the spectral density). Analog filters should be used prior to the sampling. The bandwidth of the antialiasing filter should be be somewhat smaller than the sampling frequency. For low and medium-sized frequencies the filter should have a constant gain and a phase close to zero in order not to distort the measured signal unnecessarily. For high frequencies the gain should drop quickly.
- If the experimental length T_f is fixed it may be useful to sample the record

at a high sampling rate, since more measurements from the system are then collected. No optimal sampling interval exists.

- If the total number of sample points N is fixed an optimal sampling interval exists. Then it is far worse to use a too large sampling interval than a too small one since the information losses increase rapidly when the sampling interval increases from the optimal value.
- In principle, the choice of optimal experimental length is a cost-benefit problem where the value of achieving extra information has to be balanced against the cost of obtaining the information.

5.7 References

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Chapter 6

On the Optimal Sensor Location Problem

The optimal sensor location problem (OSLP) can be stated as follows:

- Given N_s number of sensors, where should they be located in a structure so that the records obtained from those locations yield the most accurate estimates of the unknown parameters,

  or more general:
- Given an infinite number of sensors, what is the optimal number of sensors, necessary for obtaining sufficient information of the response, and where should they be located in a structure so that the records obtained from those locations yield the most accurate estimates of the unknown parameters.

In a dynamic structural identification experiment the number of sensors N_s is usually limited to minimize the cost of the instrumentation. Extensive instrumentation may be possible for a simple structure, but becomes more difficult, as the size of the structure increases due to the number and cost of the sensors and cabling required. Also in some circumstances, the time required for sensor installation may be extensive or critical. The problems of data processing, monitoring, computer storage and analysis become much harder as more sensors are used.

In the past, the OSLP has often found its solution from practical considerations, e.g. location on the antinode (problem of signal-to-noise ratio) or deck level in offshore structures (problem of cost). To improve the signal-to-noise ratio while recording the structural response, the sensors need to be located as closely as possible to the antinodes of the lower frequency mode shapes, as it has been observed from records of the structural response to e.g. wave loads that only the first few modes of vibration are significant participants in the structural vibrations. Following this strategy, an "optimal" instrumentation layout can be designed.

However, if there is more than one mode and, above all, more than one sensor, the OSLP is very difficult to solve by the antinode technique. Based on the antinode technique and experience Rojahn et al. [1] have given guidelines for placement of sensors for measuring the earthquake response of buildings.

In this chapter, several mathematical solutions to the OSLP proposed in the literature are reviewed and classified according to their main characteristics. It may be noticed that the location of sensors can be optimal in different senses, e.g. in a Fisherian sense. In section 6.1 and section 6.3 a review will be given of methods proposed to solve the OSLP for discrete and continuous vibrating systems, respectively. In section 6.2 an example is given to investigate the properties of the solution to the OSLP if it is based on the Fisher information matrix. The example is concerned with the influence of the changes in prior parameter estimates and the input characteristics of the optimal location of one sensor in a two degrees-of-freedom system. In section 6.4 an example is given where different methods to solve the OSLP for more than one sensor are investigated.

It may be noticed that not all the methods are proposed for civil engineering applications. The review seems to be a first attempt to review the methods available for solving the OSLP. It will be seen that few researchers have tried to solve the problems mentioned above. Especially, the problem of estimating the optimal number of sensors seems to be nearly unsolved.

It may be noticed that the OSLP for parameter estimation is closely related to the problem of an optimal location of sensors and controllers for control of systems which seem to be more extensively studied than the OSLP, especially for large space structures. Optimization of sensor and controller locations is important in active control of civil engineering structures according to economy and external energy savings. References considering the problem of an optimal location of sensor and controller are e.g. Norris et al. [2], Chang et al. [3], Soong et al. [4], Schulz et al. [5], Juang et al. [6], Wu et al. [7], Munak [8] and Vander Velder et al. [9]. Further, the OSLP is also related to the problem of optimal locations of sensors for failure detection of systems by vibration monitoring considered in e.g. Watanabe et al. [10] and Ferreti et al. [11].

6.1 On the OSLP for Discrete Systems

In this section different methods proposed to solve the OSLP for systems modelled as discrete systems will be considered.

6.1.1 Sensor Positioning by Using the Ibánẽz Modal Method

In the following section, a solution to the OSLP developed in Ibánẽz et al. [12] will be considered. See also Ibánẽz [13] and Ibánẽz [14]. The solution is based

on the mode shapes estimated from a priori model of the structure to be identified. The method determines an instrumentation scheme from a set of alternative instrumentation schemes to identify modal characteristics of the structure. The approach characterises the ability of alternative sensor location schemes to identify modal characteristics from outputs of sensors located at the structure. The solution of the OSLP is based on a matrix. One matrix for each alternative sensor location scheme. This matrix is the matrix product of the pseudo-inverse of the shape matrix $\overline{\overline{\Phi}}^I$ and the shape matrix $\overline{\overline{\Phi}}$. Therefore, prior to using Ibánẽz's method it is necessary to determine the eigenvectors for the vibration modes that have been specified. A detailed discussion of the properties and computation of the pseudo-inverse matrix is found in e.g. Jackson [15] and Gill et al. [16]

if the structural response may be computed by modal superposition, the measured accelerations $\ddot{\bar{y}}(t)$ are given by

$$\ddot{\bar{y}}(t) = \overline{\overline{\Phi}}\ddot{\bar{q}}(t) \tag{6.1}$$

where $\ddot{\bar{y}}(t)$ and $\ddot{\bar{q}}(t)$ are the acceleration vectors for the measured output coordinates and for the structural modal modes, respectively. The output may be measured at all or only a few coordinates in the system. Typically, the number of sensors will be less than the order of the system and less than the number of eigenvalues to be identified. If measuring is performed at N_s locations where N_s is the number of sensors and there are n modes, $\overline{\overline{\Phi}}$ is a $N_s \times n$ matrix.

The object of the technique is to estimate or isolate each modal response from a weighted sum of the measured outputs. If $N_s = n$ the inverse of the shape matrix exists and

$$\ddot{\bar{q}}(t) = \overline{\overline{\Phi}}^{-1}\ddot{\bar{y}}(t) \tag{6.2}$$

The number of sensors will normally not equal the number of modes, i.e. the inverse of the shape matrix does not exist. Then by using the pseudo-inverse of the shape matrix, which always exists, instead of the classical inverse an estimate of the modal response $\ddot{\bar{q}}^{est}(t)$ based upon the measured response can be estimated by

$$\ddot{\bar{q}}^{est}(t) \approx \overline{\overline{\Phi}}^I \ddot{\bar{y}}(t) \tag{6.3}$$

The ability of the pseudo-inverse operator to predict the modal response can be estimated by substituting equation (6.1) into equation (6.3).

$$\ddot{\bar{q}}^{est}(t) = \overline{\overline{\Phi}}^I \overline{\overline{\Phi}}\ddot{\bar{q}}(t) \tag{6.4}$$

Since the left hand side is the modal response estimated by the pseudo-inverse technique the matrix product of the pseudo-inverse shape matrix and the shape matrix

$$\overline{\overline{\Phi}}^I \overline{\overline{\Phi}} \tag{6.5}$$

should be an identity matrix. This will be true if the number of sensors is equal to the number of modes. If the product in (6.5) is very close to the identity

matrix, then according to Ibáñez et al. [12] the placement of sensors allows a good definition of the modal response. The product (6.5) may then be understood as an estimate of the quality of the modal estimates.

Using Ibáñez's method the output from an analysis of the sensor location scheme is a matrix. To reduce the amount of data required of the output from such an analysis scalar criteria functions computed from the matrix could be developed. In Ibáñez [12] the whole matrix is considered. These scalar criteria should indicate the size of off- diagonal elements in the matrix.

The method, mentioned above, does not give a dispersion measure of the parameters to be identified, i.e. the method does not make it possible to find out whether the system parameters can be estimated at all for a structural system and a given excitation by means of response measurements. The method can only be interpreted as tools to be used as an aid for the engineer in determining acceptable locations of N_s number of sensors. It is emphasised in Ibáñez [12] that the method should be used as an aid for the decision process for establishing an instrumentation scheme rather than the basis for a strict decision rule, i.e. the method must be used in combination with, and not as a substitute for, good sense.

In the following sections three different methods proposed to solve the OSLP based on the estimated parameter covariance matrix are considered.

6.1.2 Sensor Positioning by Using the Estimated Covariance Matrix Obtained by a Perturbation Approach

One of the first solutions to the OSLP based on the parameter covariance matrix seems to be given in Shah et al. [17] (1977). In brief, they used a linear relationship between small perturbations in a finite dimensional representation of the system parameters $\overline{\theta}$ to be estimated and a finite sample of observations of the system response is used to determine approximately the covariance $\overline{\overline{C}}_{\hat{\theta}_N}$ of the parameter estimates $\hat{\overline{\theta}}_N$. The so-called observations are obtained by a simulation of the model chosen based on known prior input and prior parameter estimates. The error in the parameter estimates is minimized yielding the optimal locations of the sensors such that a suitable scalar measure of the covariance matrix is minimized. It is assumed that the parameter estimate of the parameters to be identified is near their true values $\overline{\theta}_0$. Then it is possible to obtain an approximate linear relation between small variations and the simulated response. Using this relation an expression for the parameter covariance matrix representing the uncertainty of the parameter estimates can be obtained as described in the following.

It is assumed that the measuring equation for a measured displacement response at a given node is given by

$$y_i(t) = y_i(t|\overline{\theta}) + e_i(t) \qquad i = 1, 2, .., n \tag{6.6}$$

where $e_i(t)$ describes the measurement noise at location i taken as a zero mean noise process $\{\mathcal{E}(t)\}$. The word noise is used in a broad sense, as it is seen in

the foregoing section, for discrepancies between the measured response and the response obtained by the theoretical model. If the response is measured at N_s locations (6.6) becomes

$$\overline{y}(t) = \overline{y}(t, \overline{\theta}) + \overline{e}(t) \tag{6.7}$$

where $\overline{y}(t)$ is a $N_s \times 1$ observation vector. The parameter estimates $\hat{\overline{\theta}}_N$ are determined such that the response vector $\overline{y}(t, \overline{\theta})$ obtained by a model is as close as possible to the observed in a least square sense.

If the parameter estimates $\hat{\overline{\theta}}_N$ is close to the true parameter vector $\overline{\theta}_0$ then $\overline{y}(t, \overline{\theta}_0)$ can be expanded in terms of $\overline{y}(t, \hat{\overline{\theta}}_N)$. Whenever $t = t_j$ the first order expansion is

$$\overline{y}(t_j, \overline{\theta}_0) \approx \overline{y}(t_j, \hat{\overline{\theta}}_N) + \frac{\partial \overline{y}(t_j, \hat{\overline{\theta}}_N)}{\partial \hat{\overline{\theta}}_N} (\overline{\theta}_0 - \hat{\overline{\theta}}_N) \tag{6.8}$$

where $\overline{y}(t, \overline{\theta}_0)$ is a N_s dimensional vector. $\frac{\partial \overline{y}(t, \hat{\overline{\theta}}_N)}{\partial \hat{\overline{\theta}}_N}$ is a $N_s \times N_{\overline{\theta}}$ sensitivity matrix where $N_{\overline{\theta}}$ is the number of parameter in the parameter vector $\overline{\theta}$. Denoting the error $\overline{\epsilon}(t, \overline{\theta})$ at the time t_j between the measured and the estimated response

$$\overline{\epsilon}(t_j, \hat{\overline{\theta}}_N) = \overline{y}(t_j)^m - \overline{y}(t_j, \hat{\overline{\theta}}_N) \tag{6.9}$$

it is obtained from (6.8) that

$$\overline{\epsilon}(t_j, \hat{\overline{\theta}}_N) = \overline{y}(t_j, \overline{\theta}_0) + \overline{e}(t_j) - \overline{y}(t_j, \hat{\overline{\theta}}_N) \tag{6.10}$$

Denoting $\frac{\partial \overline{y}(t, \hat{\overline{\theta}}_N)}{\partial \hat{\overline{\theta}}_N}$ by $\overline{\overline{B}}_j$ following first order expansion can be written

$$\overline{\epsilon}(t_j, \hat{\overline{\theta}}_N) = \overline{e}(t_j) + \overline{\overline{B}}_j (\overline{\theta}_0 - \hat{\overline{\theta}}_N) \tag{6.11}$$

If

$$\overline{\varrho}^T = \{\overline{\epsilon}(t_1, \hat{\overline{\theta}}_N), \overline{\epsilon}(t_2, \hat{\overline{\theta}}_N), .., \overline{\epsilon}(t_N, \hat{\overline{\theta}}_N)\} \tag{6.12}$$

$$\overline{\eta}^T = \{\overline{e}(t_1), \overline{e}(t_2), .., \overline{e}(t_N)\} \tag{6.13}$$

$$\overline{\overline{A}}^T = \{\overline{\overline{B}}_1^T | \overline{\overline{B}}_2^T | ... | \overline{\overline{B}}_N^T\} \tag{6.14}$$

(6.11) can be written

$$\overline{\varrho} = \overline{\eta} + \overline{\overline{A}} (\overline{\theta}_0 - \hat{\overline{\theta}}_N) \tag{6.15}$$

N is the number of sample points.
Let $\overline{\xi} = \overline{\varrho} - \overline{\eta}$, so that

$$\overline{\overline{A}} (\overline{\theta}_0 - \hat{\overline{\theta}}_N) = \overline{\xi} \tag{6.16}$$

Since both $\overline{\varrho}$ and $\overline{\eta}$ are random vectors, so is $\overline{\xi}$. Then (6.16) implies that $(\overline{\theta}_0 - \hat{\overline{\theta}}_N)$ is a random vector with its statistics related to those of $\overline{\xi}$. Also, $\overline{\overline{A}}$ is a random matrix since it depends on the random vector $\hat{\overline{\theta}}_N$. If $\overline{\theta}_0$ is sufficiently close to $\hat{\overline{\theta}}_N$ the matrix $\overline{\overline{A}}$ can be expanded to first order as

$$\overline{\overline{A}} \approx \overline{\overline{A}}_0 + \frac{\partial \overline{\overline{A}}_0}{\partial \overline{\theta}_0}(\overline{\theta}_0 - \hat{\overline{\theta}}_N) \tag{6.17}$$

where $\overline{\overline{A}}_0$ is the sensitivity matrix evaluated at the true parameter value $\overline{\theta}_0$. Using (6.16) and (6.17)

$$\overline{\theta}_0 - \hat{\overline{\theta}}_N = \overline{\overline{A}}_0^I \overline{\xi} \tag{6.18}$$

where $\overline{\overline{A}}_0^I$ is the pseudo-inverse of the matrix $\overline{\overline{A}}_0$ The covariance of the parameter vector can then be obtained

$$\overline{\overline{C}}_{\hat{\overline{\theta}}_N} = \overline{\overline{A}}_0^I E[\overline{\xi}\overline{\xi}^T] \overline{\overline{A}}_0^{I^T} \tag{6.19}$$

It is seen that the calculated covariance depends on the pseudo-inverse matrix $\overline{\overline{A}}_0^I$ of $\overline{\overline{A}}_0$ used in the analysis.

The optimal sensor locations can now be obtained as those which minimize a scalar measure of the covariance matrix. From a practical point of view, the sensitivity matrix $\overline{\overline{A}}_0$ is unknown, since the true parameter value $\overline{\theta}_0$ about which the linearization is done is not known. Therefore, in practice, the sensitivity matrix $\overline{\overline{A}}$ computed by linearization about the estimate $\hat{\overline{\theta}}_N$ must be used. However, according to Shah et al. [17] the replacing of $\overline{\overline{A}}_0$ by $\overline{\overline{A}}$ does not cause noticeable errors in the evaluation of $\overline{\overline{C}}_{\hat{\overline{\theta}}_N}$.

In principle, the covariance matrix of $\overline{\xi}$ can only be obtained if the covariance of $\overline{\varrho}$ is available. However, the latter is not easily available prior to measuring the response. Therefore, it must either be assumed or inferred from other identification computations. $\overline{\varrho}$ depends on three factors: (1) the errors made while modelling the system mathematically, (2) the observation noise process $\{\mathcal{E}(t)\}$ and (3) the degree to which the difference between the observed response $\overline{y}(t)$ and the calculated response $\overline{y}(t, \hat{\overline{\theta}}_N)$ is minimized to obtain the parameter estimate $\hat{\overline{\theta}}_N$. If sufficient effort is spent on this minimization the first two of the aforementioned components of $\overline{\varrho}$ will mainly contribute. Further, if the modelling errors are small the statistics of $\overline{\xi}$ depends on the statistics of the observation noise process.

In the following sections it will be used that the observation noise process $\{\mathcal{E}(t)\}$ is normally modelled as a stationary Gaussian white noise process with a variance $\lambda_\mathcal{E}$. I.e. the covariance $\overline{\overline{C}}_{\mathcal{E}\mathcal{E}}$ can be written

$$C^{ij}_{\mathcal{E}\mathcal{E}} = E[\mathcal{E}_i(t_1)\mathcal{E}_j(t_2)] = \lambda_\mathcal{E}\delta_{ij}\delta(t_1 - t_2) \tag{6.20}$$

where δ_{ij} and $\delta(t_1 - t_2)$ denote the Kronecker delta function and the Dirac delta function, respectively. $E[\cdot]$ denotes expectation with respect to the probability distribution of the observation noise.

Modelling the noise as proposed in (6.20) means that the noise in each sensor and at each point of time are statistically independent with noise at any other point in the structure and time. This is generally not valid. For example if the noise is caused by an unmeasured structural excitation the observed response to the noise at one point in the structure will be correlated with other points in the structure. However, the above modelling of the noise is commonly used. However, in section 6.4 the modelling of the noise will be discussed further.

An example in Shah et al. [17] concerns the optimal location of one sensor in a building structure for identification the stiffness using response to earthquake ground motion. The example describes how the covariance results depend on the estimation of the pseudo-inverse matrix $\overset{=I}{A}$. One of the main conclusions in the paper is that when the observations are insensitive to some components of the parameters to be identified, additional information, in terms of a priori statistics, about the parameters is necessary to determine the accuracy of the parameter estimates.

In the following more direct methods to solve the OSLP will be presented which are both computationally superior and throws light on the rationale behind the optimal selection strategy.

6.1.3 Sensor Positioning by Using the Estimated Covariance Matrix of the Bayes Parameter Vector Estimate

In Sprandel [18] and Vestroni et al. [19] it is adopted that the optimal solution to the OSLP is the one giving the best value of a scalar measure of the covariance matrix of the Bayes parameter estimates.

The method proposed in Sprandel [18] is developed as a part of a system identification technique. This system identification technique has been developed to identify joint and member characteristics (moment of inertia, stiffness coefficient for joints, damping coefficient for joints etc.), from experimental time response data, in a given general user-definable finite-element model (FEM) for which joint characteristics may be explicitly modelled. It may be noticed that the purpose of the technique is not to identify modal parameters. I.e. the OSLP is solved in Sprandel [18] to find the optimal locations of N_s sensors necessary to obtain data from which the most accurate member characteristic estimates can be evaluated. This principle proposed in Sprandel [18] can also be used to solve the OSLP if the aim of the system identification is to estimate modal characteristics.

The system identification technique developed in Sprandel [18] is based on a measured time history of the excitation, either forced excitation, base motion or snap-back testing. The optimal parameter estimates are defined as a set of parameter estimates minimizing some error criteria based on the theoretical response from

the FEM due to the measured excitation and the experimental response data. In Bayes estimation it is used that the estimation scheme minimizes the expected squared error given the posterior distribution of the parameter vector which is computed from the observation vector, the theoretical model and a priori probability distributions characterizing the observation and parameter vectors.

From chapter 2 it is seen that the covariance matrix of the Bayesian parameter estimate $\hat{\bar{\theta}}$ can be written

$$\overline{\overline{C}}_{\hat{\bar{\theta}}} = \left(\overline{\overline{C}}_{\hat{\bar{\theta}}_p}^{-1} + \overline{\overline{\Delta}}^T \overline{\overline{C}}_{\varepsilon\varepsilon} \overline{\overline{\Delta}} \right)^{-1} \tag{6.21}$$

where $\overline{\overline{C}}_{\hat{\bar{\theta}}}$ and $\overline{\overline{C}}_{\hat{\bar{\theta}}_p}$ are the covariance matrices for the optimal parameter estimates and initial parameter values, respectively. $\overline{\overline{\Delta}}$ is a matrix containing the gradients of the measured response with respect to the parameters to be identified.

It is seen from (6.21) that a large observation noise implies that the measurement brings less additional information about the parameters. A large prior covariance $\overline{\overline{C}}_{\hat{\bar{\theta}}_p}$ is seen to signal that the priori parameter estimate contains little information.

In the conclusion in Shah et al. [17] it is cited that a priori statistical information about the parameter vector is necessary to determine the accuracy of the parameter vector if the measured data is insensitive to one of the identification parameters. The Bayes procedure developed in Sprandel [18] and also proposed in Vestronie et al. [19] satisfies this criterion since a priori knowledge of the distribution of the parameter vector is fundamental to the Bayes approach.

6.1.4 Sensor Positioning by Using the Fisher Information Matrix

The solution of the OSLP can be simplified if the choice of the identification algorithm is restricted to the class of efficient estimators. Generally, the solution of the OSLP requires simultaneous solution of the optimization problem and the identification problem. These two problems are uncoupled by assuming an efficient estimator where the covariance matrix is given, as mentioned before, by the inverse Fisher information matrix. Solving the OSLP based on this matrix is proposed in Udwadia [20] and Bayard et al. [21].

The Fisher information matrix can be established for a n-degree-of-freedom linear mechanical system in the following way. A n-degree-of-freedom system can be described by the equation

$$\overline{\overline{M}}\ddot{\bar{y}}(t) + \overline{\overline{C}}\dot{\bar{y}}(t) + \overline{\overline{K}}\bar{y}(t) = \bar{u}(t) \tag{6.22}$$

where $\overline{\overline{M}}$, $\overline{\overline{C}}$ and $\overline{\overline{K}}$ are the $n \times n$ mass, damping and stiffness matrices. $\bar{u}(t)$ is a $n \times 1$ load vector. The response $\bar{y}(t)$ is the solution to the second order differential equation (6.22).

A parameter vector $\overline{\theta}$ is assumed to include the various parameters related to the mass matrix, the damping matrix and the stiffness matrix, respectively, which need to be identified.

If N_s number of sensors ($N_s \leq n$) are considered the problem is to find N_s out of n responses so that they contain optimal information in a Fisherian sense of the system parameter estimate $\hat{\overline{\theta}}_N$. It is assumed that the following measuring equation for a measured displacement response at a given node

$$y_i(t) = y_i(t|\overline{\theta}) + e_i(t) \qquad i = 1, 2, .., n \tag{6.23}$$

where $e_i(t)$ again describes the measurement noise at location i taken as a zero mean non-stationary Gaussian white noise process $\{\mathcal{E}(t)\}$ with a variance $\lambda_\mathcal{E}(t)$. Thus, the covariance is given by

$$C_{\mathcal{E}\mathcal{E}}^{ij} = E[\mathcal{E}_i(t_1)\mathcal{E}_j(t_2)] = \lambda_\mathcal{E}(t_1)\delta_{ij}\delta(t_1 - t_2) \tag{6.24}$$

where δ_{ij} and $\delta(t_1 - t_2)$ denote the Kronecker delta function and the Dirac delta function, respectively.

By introducing a selection matrix $\overline{\overline{S}}$ a N_s dimensional output measurement vector $\overline{y}^m(t)$ is obtained, defined such that

$$\overline{y}(t) = \overline{\overline{S}}\overline{y}(t|\overline{\theta}) + \overline{\overline{S}}\overline{e}(t) \tag{6.25}$$

which can be written

$$\overline{y}(t) = \overline{h}_s(t) + \overline{v}(t) \tag{6.26}$$

The $N_s \times n$ matrix $\overline{\overline{S}}$ is a matrix of "1s" and "0s" such that every row has a single "1" and no column has more than a single "1". Hence, the matrix $\overline{\overline{S}}$ acts to select elements, i.e. location of sensors, from $\overline{y}(t|\overline{\theta})$

The Fisher information matrix $\overline{\overline{J}}$, in continuous time, associated with identification of $\overline{\theta}$ for the system in (6.21) using the measurement vector $\overline{y}(t)$ in (6.26) is given as follows for a measuring time T_f

$$\overline{\overline{J}} = \int_0^{T_f} \left(\frac{\partial \overline{h}_s(t)}{\partial \overline{\theta}}\right)^T \left(\frac{\partial \overline{h}_s(t)}{\partial \overline{\theta}}\right) \lambda_\mathcal{E}^{-1}(t)dt \tag{6.27}$$

It may be noticed that a discrete in time formulation of the Fisher information matrix could also have been used. Introducing (6.25) in equation (6.27) the following can be written

$$\overline{\overline{J}} = \int_0^{T_f} \left(\frac{\partial \overline{y}(t)}{\partial \overline{\theta}}\right)^T \overline{\overline{S}}^T \overline{\overline{S}} \left(\frac{\partial \overline{y}(t)}{\partial \overline{\theta}}\right) \lambda_\mathcal{E}^{-1}(t)dt \tag{6.28}$$

where the ij element of $\frac{\partial \overline{y}(t)}{\partial \overline{\theta}}$ can be written

$$\left[\frac{\partial \overline{y}(t)}{\partial \overline{\theta}}\right]_{ij} = \frac{\partial y_i(t)}{\partial \theta_j} \qquad i = 1, 2, .., n; j = 1, 2, .., n_{\overline{\theta}} \tag{6.29}$$

The Fisher information matrix is seen to be symmetric and it depends on the length of the record available, as well as the locations of the sensors as determined by the matrix product $\overline{\overline{S}}^T\overline{\overline{S}}$.

If the N_s locations, where the sensors are to be placed, are denoted $z_k, k = 1,2,..,N_s$ then

$$\overline{\overline{P}} = \overline{\overline{S}}^T\overline{\overline{S}} = \sum_{k=1}^{N_s} \overline{\overline{I}}_{z_k} \tag{6.30}$$

where the $n \times n$ diagonal matrix $\overline{\overline{I}}_{z_k}$ has all its elements equal to zero except the elements of the z_k row which is unity. Using $\overline{\overline{P}}$ the ij element of $\overline{\overline{J}}$ can be written

$$J^{ij} = \sum_{k=1}^{N_s} \int_0^{T_f} \left[\frac{\partial y_{z_k}(t)}{\partial \theta_i} \frac{\partial y_{z_k}(t)}{\partial \theta_j} \lambda_{\mathcal{E}}^{-1}(t) \right] dt \tag{6.31}$$

Each element of J^{ij} represents the cross-sensitivity of measurement with respect to the response $y_{z_k}(t)$ of node with the location z_k.

The optimal sensor locations can now be obtained by picking N_s locations $z_k, k = 1,2,..,N_s$ out of n so that a scalar measure of the matrix $\overline{\overline{J}}^{-1}$ is minimized.

6.1.5 Sensor Positioning for Lightly Damped Structures by using the Determinant of the Inverse of the Fisher Information Matrix

Above in section 6.1.4 it was shown that the OSLP can be uncoupled from the choice of identification algorithm if an asymptotically efficient estimator is chosen. If it is assumed that the structure under consideration is lightly damped and a D-optimality design criterion is used, then the OSLP, or experimental design, can be further simplified, see Bayard et. al. [21].

It is assumed that the parameters to be identified are the modal frequencies and damping ratios in a lightly damped n degrees-of-freedom system. I.e. the parameter vector is given by

$$\overline{\theta} = [\omega_1, \zeta_1, .., \omega_n, \zeta_n]^T \tag{6.32}$$

where ω_1 and ζ_1 are the angular frequency and damping ratio of the first mode, respectively.

Using the modal approach, see e.g. Thomsen [22], the measuring equation can be written

$$y(t)_i = \overline{r}_i^T \overline{\overline{\Phi}} \overline{q}(t) + e_i(t) \qquad i = 1,2,..,n \tag{6.33}$$

$e_i(t)$ describes the measurement noise at location i taken as a zero mean stationary Gaussian white noise process $\{\mathcal{E}(t)\}$ with the variance $\lambda_{\mathcal{E}}$. Thus, the covariance is given by

$$\overline{\overline{C}}_{\mathcal{E}\mathcal{E}} = E[\mathcal{E}_i(t_1)\mathcal{E}_j(t_2)] = \lambda_{\mathcal{E}} \delta_{ij} \delta(t_1 - t_2) \tag{6.34}$$

where δ_{ij} and $\delta(t_1 - t_2)$ denote the Kronecker delta function and the Dirac delta function, respectively. $\overline{r}_i$ is the vector associated with the location of the ith sensor. It is assumed that the sensors can be placed in each node and that each sensor measures the same kind of response. $\overline{\overline{\Phi}}$ is a mode shape matrix including the mode shapes $\overline{\phi}_i$. $\overline{q}(t)$ is a $n \times 1$ vector including the modal response. $\overline{q}(t)$ is the solution to the second order differential equation

$$\ddot{\overline{q}}(t) + 2\overline{\overline{\Omega}}\,\overline{\overline{D}}\,\dot{\overline{q}}(t) + \overline{\overline{\Omega}}^2\,\overline{q}(t) = \overline{\overline{\Phi}}^T\overline{\overline{B}}\,\overline{u}(t) \tag{6.35}$$

where $\overline{\overline{B}}$ is an input matrix determined by the location of the input signal $\overline{u}(t)$. $\overline{\overline{D}}$ and $\overline{\overline{\Omega}}$ are diagonal matrices containing the modal damping and frequencies, respectively.

By introducing a selection matrix similar to the discussion in section 6.1.4 the Fisher information matrix associated with identification of $\overline{\theta}$ in (6.32) is given for an infinite-time horizon as follows

$$\overline{\overline{J}} = \sum_{i=1}^{N_s} \frac{\Xi_i}{\lambda_i} \int_0^\infty \left(\frac{\partial \overline{q}(t)}{\partial \overline{\theta}}\right)^T \overline{\overline{\Phi}}^T \overline{r}_i \overline{r}_i^T \overline{\overline{\Phi}} \left(\frac{\partial \overline{q}(t)}{\partial \overline{\theta}}\right) dt \tag{6.36}$$

where $\Xi_i = 1$ if $y_i(t)$ is assumed to be measured, otherwise $\Xi_i = 0$. and

$$\frac{\partial \overline{q}(t)}{\partial \overline{\theta}} = \begin{bmatrix} \frac{\partial q_1(t)}{\partial \omega_1} & \frac{\partial q_1(t)}{\partial \zeta_1} & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \frac{\partial q_2(t)}{\partial \omega_2} & \frac{\partial q_2(t)}{\partial \zeta_2} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & \frac{\partial q_n(t)}{\partial \omega_n} & \frac{\partial q_n(t)}{\partial \zeta_n} \end{bmatrix} \tag{6.37}$$

The block structure of (6.37) follows since

$$\frac{\partial q_i(t)}{\partial \omega_j} = \frac{\partial q_i(t)}{\partial \zeta_j} = 0 \quad \forall: \quad i \neq j \tag{6.38}$$

Substituting (6.37) into (6.36) yields the expression for $\overline{\overline{J}}$

$$\overline{\overline{J}} = \begin{bmatrix} \overline{\overline{J}}^{11} & \overline{\overline{J}}^{12} & \cdot & \cdot & \overline{\overline{J}}^{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \overline{\overline{J}}^{n1} & \overline{\overline{J}}^{n2} & \cdot & \cdot & \overline{\overline{J}}^{nn} \end{bmatrix}$$

where

$$J^{ij} = \begin{bmatrix} F^{ij}_{\omega\omega} & F^{ij}_{\omega\zeta} \\ F^{ij}_{\omega\zeta} & F^{ij}_{\zeta\zeta} \end{bmatrix} \tag{6.39}$$

$$F^{ij}_{\omega\omega} = S_{ij}(\overline{\Xi}) \int_0^\infty \frac{\partial q_i(t)}{\partial \omega_j} \frac{\partial q_j(t)}{\partial \omega_j} dt \tag{6.40}$$

$$F^{ij}_{\omega\zeta} = F^{ij}_{\zeta\omega} = S_{ij}(\overline{\Xi}) \int_0^\infty \frac{\partial q_i(t)}{\partial \omega_j} \frac{\partial q_j(t)}{\partial \zeta_j} dt \tag{6.41}$$

$$F_{\zeta\zeta}^{ij} = S_{ij}(\overline{\Xi}) \int_0^\infty \frac{\partial q_i(t)}{\partial \zeta_j} \frac{\partial q_j(t)}{\partial \zeta_j} dt \tag{6.42}$$

$$\overline{\Xi} = [\Xi_1, \Xi_2, ..\Xi_{N_s}]^T \tag{6.43}$$

$$S_{ij}(\overline{\Xi}) = \sum_{k=1}^{N_s} \frac{\Xi_k}{\lambda_\varepsilon^k} (\overline{r}_k^T \phi_i)(\overline{r}_k^T \phi_j) \tag{6.44}$$

By using the Parseval theorem, see e.g. Brigham [23], the integral in (6.40), (6.41) and (6.42) can be written in the frequency domain

$$F_{\omega\omega}^{ij} = S_{ij}(\overline{\Xi}) J_{\omega\omega}^{ij}(\overline{u}) \tag{6.45}$$

$$F_{\omega\zeta}^{ij} = F_{\zeta\omega}^{ij} = S_{ij}(\overline{\Xi}) J_{\omega\zeta}^{ij}(\overline{u}) \tag{6.46}$$

$$F_{\zeta\zeta}^{ij} = S_{ij}(\overline{\Xi}) J_{\zeta\zeta}^{ij}(\overline{u}) \tag{6.47}$$

where

$$J_{\omega\omega}^{ij}(\overline{u}) = \frac{(-1)}{2\pi\imath} \int_{-\imath\omega}^{\imath\omega} s^2 G_i^\omega(s) G_i^\omega(s) \overline{\phi}_i^T \overline{\overline{B}} \tilde{\overline{u}}(s) \overline{\phi}_j^T \overline{\overline{B}} \tilde{\overline{u}}(-s) ds \tag{6.48}$$

$$J_{\omega\zeta}^{ij}(\overline{u}) = \frac{(-1)}{2\pi\imath} \int_{-\imath\omega}^{\imath\omega} s^2 G_i^\omega(s) G_i^\zeta(s) \overline{\phi}_i^T \overline{\overline{B}} \tilde{\overline{u}}(s) \overline{\phi}_j^T \overline{\overline{B}} \tilde{\overline{u}}(-s) ds \tag{6.49}$$

$$J_{\zeta\zeta}^{ij}(\overline{u}) = \frac{(-1)}{2\pi\imath} \int_{-\imath\omega}^{\imath\omega} s^2 G_i^\zeta(s) G_i^\zeta(s) \overline{\phi}_i^T \overline{\overline{B}} \tilde{\overline{u}}(s) \overline{\phi}_j^T \overline{\overline{B}} \tilde{\overline{u}}(-s) ds \tag{6.50}$$

where $\tilde{\overline{u}}(s) = \mathcal{L}\{\overline{u}(t)\}$ is the Laplace transformation.

$$G_i^\omega(s) = \frac{2\zeta_i s + 2\omega_i}{(s^2 + 2\zeta_i \omega_i s + \omega_i^2)^2} \tag{6.51}$$

$$G_i^\zeta(s) = \frac{2\omega_i s}{(s^2 + 2\zeta_i \omega_i s + \omega_i^2)^2} \tag{6.52}$$

Based on these expressions for the Fisher Information Matrix in the frequency domain it is shown in Bayard et al. [21] that the Fisher information matrix (normalized in a suitable fashion) tends towards a block diagonal matrix for lightly damped structures. It is assumed that the input energy E_u is constrained, i.e.

$$E_u = \int_0^\infty \overline{u}^T(t) \overline{u}(t) dt < \infty \tag{6.53}$$

Further, the resonance frequencies are assumed to be distinguished, i.e.

$$\omega_i \neq \omega_j \quad \forall : \quad i, j; i \neq j \tag{6.54}$$

To study the properties of a structure as the damping gets small a system damping parameter ζ and normalized damping is introduced

$$\zeta = \max_i \zeta_i \tag{6.55}$$

$$d_i = \frac{\zeta_i}{\zeta} \qquad i = 1, 2, .., n \tag{6.56}$$

By using these quantities it can be shown that $F^{ij}_{\omega\omega}$ and $F^{ij}_{\zeta\zeta}$ grows at least as fast as ζ^{-3} as $\zeta \to 0$. Therefore, an appropriate normalization of $\overline{\overline{J}}$ is given by $\overline{\overline{J}}_\zeta$, where

$$\overline{\overline{J}}_\zeta = \zeta^3 \overline{\overline{J}} \tag{6.57}$$

Based on the above assumptions an important result concerning the behaviour of $\overline{\overline{J}}_\zeta$ for $(\zeta << 1)$ is given by

$$\det \overline{\overline{J}}_\zeta = \det \hat{\overline{\overline{J}}} + \mathcal{O}(\zeta^2) \qquad \zeta \to 0 \tag{6.58}$$

where the determinant det of the block-diagonal matrix $\hat{\overline{\overline{J}}}$ is

$$\det \hat{\overline{\overline{J}}} = diag[\zeta^3 J^{11}, .., \zeta^3 J^{nn}] \tag{6.59}$$

$\mathcal{O}(\zeta^2)$ indicates an error or order (ζ^2), i.e. for a lightly damped structure $(\zeta << 1)$ (6.58) represents a meaningful approximation.

If a D-optimality design criterion is selected the optimal experimental design is given by

$$\begin{aligned}
\max_{\overline{\Xi}, \overline{u}(t)} \det \overline{\overline{J}}_\zeta &= \max_{\overline{\Xi}, \overline{u}(t)} \det \hat{\overline{\overline{J}}} + \mathcal{O}(\zeta^2) \\
&= \max_{\overline{\Xi}, \overline{u}(t)} \prod_{i=1}^{n} \det \hat{\overline{\overline{J}}}^{ii}_\zeta + \mathcal{O}(\zeta^2) \\
&= \max_{\overline{\Xi}} \left(\prod_{i=1}^{n} S^2_{ii}(\overline{\Xi}) \right) \max_{\overline{u}(t)} \prod_{i=1}^{n} \left(\det \overline{\overline{J}}^{ii}(\overline{u}) \right) \\
&+ \mathcal{O}(\zeta^2)
\end{aligned} \tag{6.60}$$

It is seen from (6.60) that the light damping approximation has a separation effect on the design of the experiment, i.e. the optimal input design $\overline{u}^*$ and the optimal sensor placement vector $\overline{\Xi}^*$ can be solved independently according to the following decouple criteria

$$\overline{u}^* = \max_{\overline{u}} \prod_{i=1}^{n} \det \overline{\overline{J}}^{ii}(\overline{u}) \tag{6.61}$$

$$\overline{\Xi}^* = \max_{\overline{\Xi}} \left(\prod_{i=1}^{n} S_{ii}(\overline{\Xi}) \right) \tag{6.62}$$

The resulting design $(\overline{u}^*, \overline{\Xi}^*)$ is then optimal, within $\mathcal{O}(\zeta^2)$ for the original D-optimality. The decoupling effect is seen to give a significant simplification of the experimental design. Thus, the optimal input design and the OSLP can be solved

independently. Moreover, the decoupling results indicate that for lightly damped structures the sensors can be placed optimally within $\mathcal{O}(\delta^2)$ by utilizing mode shape information only. This means that the values of actual modal frequencies and damping ratios are not required for sensor placement. This is significant since sensors can be placed roughly based on mode shape information which is less uncertain prior to the measurements than information about the modal frequencies and damping ratios.

In section 6.2 an example is given to investigate the properties of the solution to the OSLP, if the Fisher information matrix is used. The example considers the influence of the changes in prior parameter estimates and the input characteristics on the optimal sensor location.

6.2 Example 6.1: Optimal Location of One Sensor to Variations of Prior Parameter Estimates and Input Characteristics

A 2DOF linear mechanical structure which is subjected to a base excitation is considered, see figure 6.1.

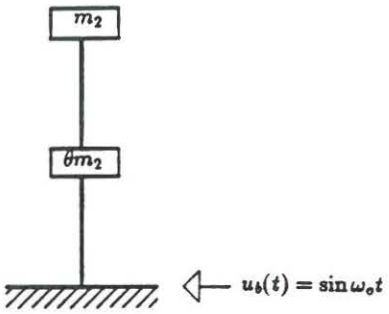

Figure 6.1: 2DOF linear mechanical structure which is subjected to a base excitation.

The dynamic model for the structure is given by

$$\overline{\overline{M}}\ddot{\overline{y}}(t) + \overline{\overline{C}}\dot{\overline{y}}(t) + \overline{\overline{K}}\overline{y}(t) = \overline{u}(t) \tag{6.63}$$

where $\overline{\overline{M}}$, $\overline{\overline{C}}$ and $\overline{\overline{K}}$ are the 2×2 mass, damping and stiffness matrices. $\overline{\overline{u}}(t)$ is a 2×1 load vector. The response $\overline{y}(t)$ is the solution to the second order differential equation (6.63).

It is assumed that the mass matrix $\overline{\overline{M}}$, the damping matrix $\overline{\overline{C}}$ and the stiffness matrix $\overline{\overline{K}}$ are given by

$$\overline{\overline{M}} = \begin{bmatrix} \theta m_2 & 0 \\ 0 & m_2 \end{bmatrix} \tag{6.64}$$

$$\overline{\overline{C}} = c\overline{\overline{K}} \tag{6.65}$$

$$\overline{\overline{K}} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \tag{6.66}$$

where m_2 is the mass of the upper mass and k is the stiffness of the springs.
Since an excitation of the base is applied (6.63) can be written

$$\overline{\overline{M}}\ddot{\overline{y}}(t) + \overline{\overline{C}}\dot{\overline{y}}(t) + \overline{\overline{K}}\overline{y}(t) = -\overline{\overline{M}}\,\overline{\overline{I}}\ddot{u}_b(t) \tag{6.67}$$

where $\overline{\overline{I}}$ is the identity matrix. It may be noticed that $y(t)$ does not represent the total displacement but the relative displacements of the system to the displacements of the base. $\ddot{u}_b(t)$ is the acceleration of the base. In this example a sinusoidal base excitation is considered. I.e.

$$u_b(t) = \sin\omega_o t \tag{6.68}$$

where ω_o is the frequency of the excitation and t is a time parameter.
The particular solution of (6.68) giving the stationary response $\overline{y}(t)$ when $u_b(t)$ is a sinusiodal excitation is assumed to be

$$\overline{y}(t) = Re(\overline{\overline{X}}(\omega)e^{\imath\omega_0 t}) \tag{6.69}$$

$Re(\cdot)$ denotes the real part of (6.69). Substituting this solution into (6.67), it is obtained that (6.67) can be written

$$\overline{y}(t) = Re(\overline{\overline{H}}(\omega)\overline{F}e^{\imath\omega_0 t}) \tag{6.70}$$

where

$$\overline{\overline{H}}(\omega) = \left[\overline{\overline{K}} - \omega^2\overline{\overline{M}} + \imath\omega\overline{\overline{C}}\right]^{-1} \tag{6.71}$$

and

$$\overline{F} = \omega_0^2\overline{\overline{M}}\,\overline{\overline{I}} \tag{6.72}$$

$\overline{\overline{H}}(\omega)$ is the frequency response matrix which is generally complex.
It is assumed that one sensor is used and that the measurements are given by the following measuring equation

$$y_i(t) = y_i(t,\overline{\theta}) + e_i(t) \qquad i = 1,2 \tag{6.73}$$

where $e_i(t)$ describes the measurement noise at location i taken as a zero mean stationary Gaussian white noise process $\{\mathcal{E}(t)\}$ with the variance $\lambda_\mathcal{E}$.
The parameter assumed to be identified is the mass ratio θ of the lower mass to the upper mass. Since only one parameter is being estimated the Fisher information

matrix reduces to a scalar. If the response is measured at the lower mass the Fisher information matrix becomes

$$J_1 = \int_0^{T_f} \left(\frac{\partial y_1(t)}{\partial \theta} \right)^2 \lambda_\varepsilon^{-1} dt \tag{6.74}$$

and if the response is measured at the upper mass the Fisher Information matrix becomes

$$J_2 = \int_0^{T_f} \left(\frac{\partial y_2(t)}{\partial \theta} \right)^2 \lambda_\varepsilon^{-1} dt \tag{6.75}$$

I.e. that the sensor should be placed at the lower mass if $J_1 > J_2$. T_f is the duration time of the measurement.

In figure 6.2 the ratio of the information matrices J_1/J_2 to various values of the excitation frequency ω_0 is shown for the following quantities: $T_f = 45$ sec, $\theta = 1$, $m_2 = 1$, $k = 1$ and $c = \{0.01, 0.1, 0.2, 0.3, 0.4\}$.

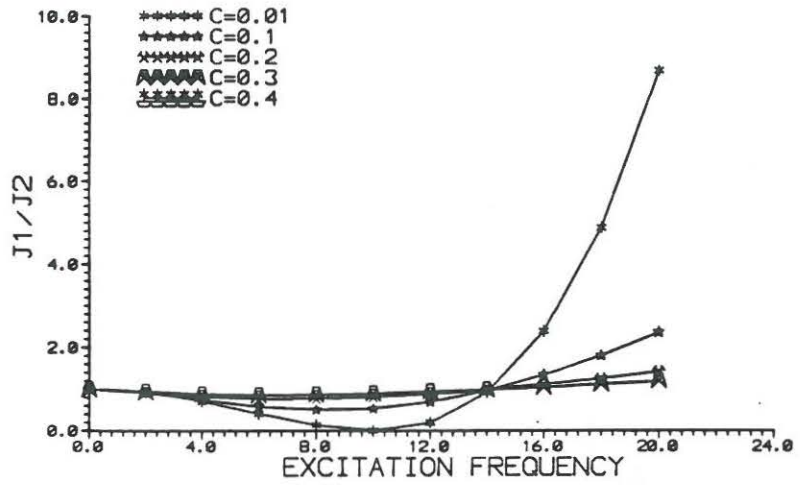

Figure 6.2: Variation of J_1/J_2 for various values of the excitation frequency ω_0.

Points on the graph with ordinates greater than unity indicate the optimal sensor location to be the lower mass level and vice versa. The graphs show that when the excitation frequency ω_0 varies the optimal sensor location changes. Further, it is seen that the responses at the two mass levels yield identical amounts of information of θ when $\omega_0 \to 0$ and when ω_0 is near the second angular eigenfrequency $\omega_2 = 16.18 rad/\text{sec}$ ($\omega_1 = 6.18 rad/\text{sec}$) as indicated by the values of $J_1/J_2 = 1$ at these frequencies. It is also seen that the same amount of information of θ will be obtained if the damping of the system is large . When the damping is small it is seen that the upper mass level is a far better location for a sensor when estimating θ for an excitation with a frequency less than the second angular eigenfrequency,

frequency larger than the second angular eigen-frequency. This means that it is more important to place the sensor optimally for a structure which is lightly damped than for a structure which is not lightly damped.

Figure 6.3 shows the optimal excitation frequency to variations of the values of the parameter to be identified. The corresponding optimal location of one sensor can be seen in figure 6.4, where the ratio J_1/J_2 is shown for variations of the values of the parameter to be identified θ. It may be noticed, as seen from (6.74) and (6.75), that the information increases when T increases.

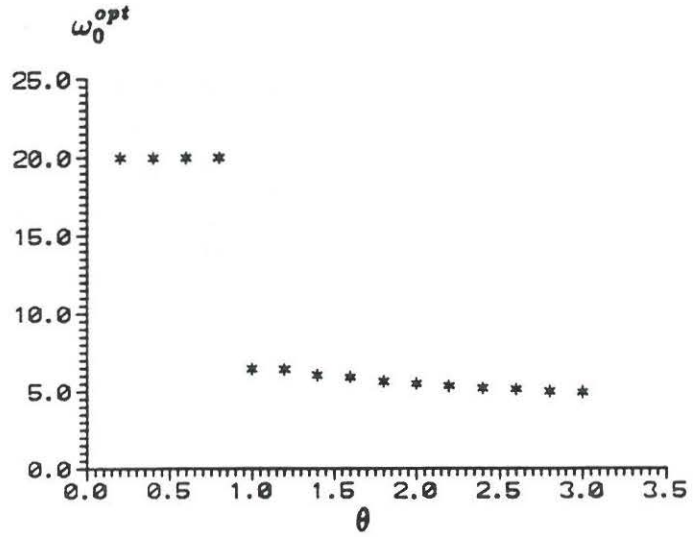

Figure 6.3: The optimal excitation frequency shown for various values of the parameter θ.

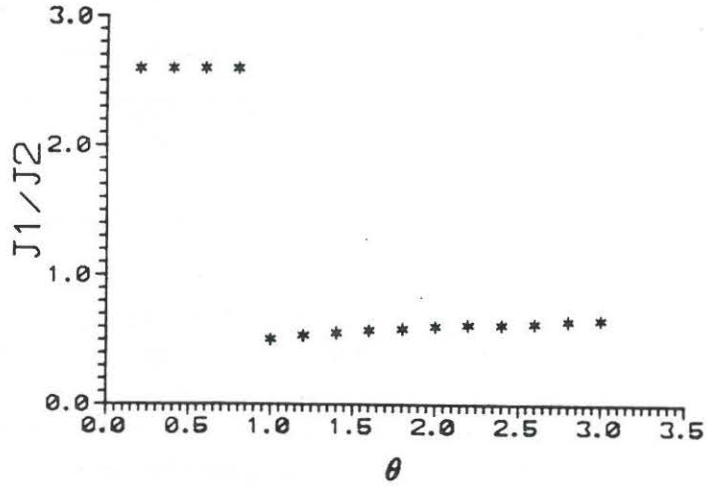

Figure 6.4: The ratio J_1/J_2, corresponding to the optimal excitation frequency, shown for various values of the parameter θ.

From the results shown above, it is, interestingly enough, noted that the optimal sensor location for estimation of θ actually depends not only on the actual values of c, representing a parameter not to be identified, but also on the value of the parameter θ itself which is to be identified. Thus, as experienced in earlier examples, it is necessary to have some a priori information about the system parameters in order to be able to ascertain the optimal sensor location. Further, it was seen that the optimal sensor location depends on the excitation. Here only one excitation type is considered, but it may be expected that the optimal sensor location also depends on the excitation type.

The conclusion of the example is that design of an experiment on purely heuristic grounds may be difficult since the simple example considered here has yielded results showing that the optimal sensor location appears to depend in a more or less complex manner on the actual parameter values of the system and the excitation.

6.3 On the OSLP for Continuous Systems

In this section, different methods proposed for solving the OSLP for dynamic systems modelled by partial differential equations, i.e distributed parameter systems, will be presented. In this chapter the methods are investigated in relation to structural systems. It may be noticed that some methods given in section 6.1 will also apply if continuous systems are considered and in the same way methods given for continuous systems in this section will also apply to discrete systems.

To identify parameter distributed systems the best thing to do is to place an infinite number of sensors, distributed in the whole spatial domain. However, this ideal case is unrealistic and it is worthwhile to ensure optimal locations of a minimal number of sensors. Only a few papers have appeared specifically on the OSLP for parameter distributed mechanical systems where optimal location strategies have been proposed. Beyond the papers mentioned in the following sections the OSLP for parameter distributed systems has also been considered in Le Pourhiet et al. [24]. The basic idea in that paper is to place sensors in a distributed parameter system described by the diffusion equation such that the identification error sensitivity with respect to the location of a new sensor is maximized. When a new sensor location has been chosen the distribution of the previous sensors is not changed afterwards. So Le Pourhiet et al. [24] do not really deal with a method for optimal location of sensors.

6.3.1 Sensor Positioning by Using the Fisher Information Matrix

Based on the Fisher information matrix Qureshi et al. [25], Rafajlowicz [26] and Rafajlowicz [27] present methods for distributed parameter systems for design of optimal experiments, and with that solutions to the OSLP. However, in Qureshi et al. [25] the Fisher information matrix is associated with the system parameters while Rafajlowicz [26] and Rafajlowicz [27] consider the system eigenvalues, i.e. parameters representing the natural frequencies of the undamped system.

If a 1-dimensional system, in the z-direction, described by a partial differential equation, is considered the following measuring equation for a measured displacement response $y^m(z_i,t)$ can be written

$$y^m(z_i,t) = y(z_i,t) + e(z_i,t) \tag{6.76}$$

where $y(z_i,t)$ is given by a chosen model. $e(z_i,t)$ describes the measuring noise at location z_i taken as a zero mean non-stationary Gaussian white noise process $\{\mathcal{E}(z,t)\}$, both in the space and the time parameters, with a variance $\lambda_\mathcal{E}(t)$, i.e. the element $C^{ij}_{\mathcal{E}\mathcal{E}}$ of the covariance matrix of the noise $\overline{\overline{C}}_{\mathcal{E}\mathcal{E}}$ is given by

$$C^{ij}_{\mathcal{E}\mathcal{E}} = E[\mathcal{E}(z_i,t_i)\mathcal{E}(z_j,t_j)] = \lambda_\mathcal{E}(t_j)\delta(z_i - z_j)\delta(t_i - t_j) \tag{6.77}$$

where $\delta(z_i - z_j)$ and $\delta(t_i - t_j)$, respectively, denote Dirac delta functions.

The Fisher information matrix $\overline{\overline{J}}$, in continuous time, associated with identification of the parameters in the parameter vector $\overline{\theta}$ associated with a system described by a partial differential equation is given as follows for a measuring time T_f

$$J^{ij} = \sum_{k=1}^{N_s} \int_0^{T_f} \frac{\partial y(z_k,t)}{\partial \theta_i} \frac{\partial y(z_k,t)}{\partial \theta_j} \lambda_\varepsilon^{-1}(t) dt \tag{6.78}$$

Each element of J^{ij} represents the cross-sensitivity of a measurement with respect to the response $y(z_k,t)$ at the location z_k.

Qureshi et al. [25], Rafajlowicz [26] and Rafajlowicz [27] assume that it is not necessary that $z_i \neq z_j$ for $i \neq j$. Further, the description of the noise (6.77) is required to hold even if $z_i = z_j, .., z_l$ for $i \neq j, i \neq l, ..., j \neq l$ etc. This means that the measuring noises are assumed to be produced independently by sensors placed very close to each other. The applicability of this assumption when continuous mechanical systems are considered will be discussed in an example given in section 6.4.

6.3.2 Sensor Positioning by Using the Concept of Entropy

In the previous section it is explained how the Fisher information matrix is proposed to be used to determine the optimal locations of sensors in the design of experiments for parametric identification. The Fisher information matrix gives a measure of the amount of information supplied by data about the unknown parameters. This measure of "information" in technical sense was for the first time used in 1925, and was introduced especially for the theory of statistical estimation, see Fisher [28]. However, the subject of information theory was originated after the second world war by Shannon. Like several other branches of mathematics, the information theory is of physical origin. It was initiated by communication scientists who studied the statistical nature of electrical communication equipment. Or more precisely, they were motivated by the problem of describing the information contents of signals in communication systems. Because concepts from information theory are so effective in communication theory is the reason why the information theory is often considered to be synonymous with communication theory, see e.g. Kullback [28] and Reza [29]. It also seems to be the fact that relatively few conclusive results have been obtained in other fields with use of the information theoretical approach. However, as mentioned in Kullback [28] the information theory is a branch of mathematical theory and mathematical statistics. Thus information theory provides an alternative approach to statistical interference and its concepts and methods are applicable to analysis of various physical and engineering systems. The information theory is especially relevant to any probabilistic system of observations since the aim of getting statistical observations is to obtain information, i.e. one could expect that the information theory is also applicable when design of parametric identification experiments is the subject.

In this section, a method to solve the problem of optimal design of experiment in

stochastic dynamics of engineering systems, see Sobczyk [30], based on the concept of entropy, is described. The concept of entropy is a measure used in information theory to measure the amount of uncertainty of real random phenomena. The entropy H_Y associated with a continuous random variable Y is given by

$$H_Y(y) = -\int_a^b f_Y(y) \log f_Y(y) dy \tag{6.79}$$

It is seen that the entropy is a function of the probability density function $f_Y(y)$. In Sobczyk [30] it is postulated that the optimal locations of N_s sensors are the locations providing maximum entropy.

If a continuous mechanical system is considered with a displacement field $y(z,t)$ of zero mean, measured at N_s points, the criterion of maximum of entropy gives

$$\max_{\overline{z}} H_{\overline{Y}}(\overline{y}) = \max_{\overline{z}} \left(-\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\overline{Y}}(\overline{y}) \log f_{\overline{Y}}(\overline{y}) d\overline{y} \right) \tag{6.80}$$

where $\overline{z} = \{z_1, ..., z_{N_s}\}$ is given by a vector of dimension equal to the number of measuring points.

If the measured displacement field is assumed to be zero mean Gaussian the entropy can be expressed in terms of the correlation function of the displacement field

$$H_{\overline{Y}}(\overline{y}) = \log \sqrt{(2\pi e)^{N_s} |\overline{\overline{R}}|} \tag{6.81}$$

where $|\overline{\overline{R}}|$ is the determinant of the matrix $\overline{\overline{R}}$ whose elements are given by the correlation function of the displacement field.

It is seen that the advantage of the above criterion (6.80) is that it can easily be generalised to the case of many observation points. Further, it is also seen that it does not depend on the parameters to be estimated from the measurements. In the example in section 6.4 it will be shown that design of parametric identification experiments based on (6.80) in some circumstances gives the same result as when a method depending on the parameters is used. However, it is not the same as assuming that the concept of entropy can be used to design experiments for parametric identification of structural systems.

6.3.3 Sensor Positioning by Using a Best Linear Unbiased Estimator

The method, see Kazimierczyk [31] and [32], which will be considered in this section, is proposed to solve the OSLP, when the aim of the experiment is parametric identification of continuous mechanical systems subjected to random load, where the response can be given, by a model, which is linear in the parameters. The method is developed to take into account that random loads acting on a structure, in general, implying random response of the generally deterministic structure. The

stochastic nature of the response of mechanical structures leads to the necessity of the use of models for experiment design where the observed quantity is a random field, both space and time dependent and correlated. This problem will be considered closer in an example given in section 6.4. Here only the method will be described.

The method is applicable if the aim of the experiment design is to determine the optimal measuring points in the problem of estimation of parameters $\overline{\theta}$ entering the following model of the measured random field

$$y^m(z_i,t) = \overline{\theta}^T \overline{f}(z_i,t) + e_1(z_i,t) \tag{6.82}$$

where $\overline{f}(z_i,t)$ is a known vector-valued function of z and t. $e_1(z_i,t)$ is a realization of the random part $\{\mathcal{E}_1(z,t)\}$ of the measured outcome describing both the measuring errors and the random fluctuations of the state of the object. It may be noticed that the random part $\{\mathcal{E}_1(z,t)\}$ is different from the random part $\{\mathcal{E}(z,t)\}$ normally modelled as a zero mean white noise process $\{\mathcal{E}(z,t)\}$ both in space and time with a variance $\lambda_{\mathcal{E}}$. If a continuous mechanical structure is subjected to random loading this assumption concerning the noise cannot be satisfied any longer since the measurements can hardly be modelled by independent random variables, i.e. in general both space and time dependence and correlation have to be taken into account.

For simplicity it is assumed that the observations are taken only once at each measuring point. Then instead of the assumption (6.77) regarding $\{\mathcal{E}(z,t)\}$ the following modelling of the elements $C^{ij}_{\mathcal{E}_1\mathcal{E}_1}$ in the covariance matrix $\overline{\overline{C}}_{\mathcal{E}_1\mathcal{E}_1}$ of the zero mean noise process can be used

$$C^{ij}_{\mathcal{E}_1\mathcal{E}_1} = E[\mathcal{E}_1(z_i)\mathcal{E}_1(z_j)] = R_{YY}(z_i,z_j;0) = E[Y(z_i)Y(z_j)] \quad \forall : i,j = 1,..,N_s \tag{6.83}$$

By using the modelling of the noise in (6.83) the best linear unbiased estimator (BLUE) $\hat{\overline{\theta}}_N$, for $\overline{\theta}$, defined as the estimator that has minimum variance among the class of all linear unbiased estimators, is given by

$$\hat{\overline{\theta}}_N = (\overline{F}^T \overline{\overline{R}}^{-1} \overline{F})^{-1} \overline{F}^T \overline{\overline{R}}^{-1} \overline{Y} \tag{6.84}$$

where

$$\overline{F} = \{\overline{f}(z_1), \overline{f}(z_2), .., \overline{f}(z_{N_s})\}^T \tag{6.85}$$

$$\overline{Y} = \{\overline{y}(z_1), \overline{y}(z_2), .., \overline{y}(z_{N_s})\}^T \tag{6.86}$$

The BLUE estimator is also the estimator minimizing the following weighted sum of squares V_N

$$V_N = (\overline{Y} - \overline{F}^T\overline{\theta})^T \overline{\overline{W}} (\overline{Y} - \overline{F}^T\overline{\theta}) \tag{6.87}$$

(6.87) corresponds to a weighted least squares estimation where the weighting matrix $\overline{\overline{W}}$ is given by the correlation matrix

$$\overline{\overline{W}} = \overline{\overline{R}}^{-1} \tag{6.88}$$

From (6.87) it is seen that the assumption in (6.83) corresponds to the correlation matrix $\overline{\overline{R}}$ of the response is expressing the relative precision of the measurements. The covariance of $\hat{\overline{\theta}}_N$ described in (6.84) can be estimated by

$$\begin{aligned}\overline{\overline{C}}_{\hat{\overline{\theta}}_N} &= E[(\hat{\overline{\theta}}_N - \overline{\theta})(\hat{\overline{\theta}}_N - \overline{\theta})^T] = (\overline{F}^T \overline{\overline{W}}\, \overline{F})^{-1} \overline{F}^T \overline{\overline{W}}\, \overline{\overline{R}}\, \overline{\overline{W}}\, \overline{F} (\overline{F}^T \overline{\overline{W}}\, \overline{F})^{-1} \\ &= (\overline{F}^T \overline{\overline{R}}^{-1} \overline{F})^{-1} \end{aligned} \tag{6.89}$$

Now, it is seen that the problem of optimal experiment design can be reduced to the common problem of minimisation of a scalar measure of the parameter covariance matrix if the parameters to be identified can be written by a model linear in the parameters.

6.4 Example 6.2: Evaluation of Different Methods to Solve the OSLP for Continuous Systems

In the previous sections different proposed solutions of the OSLP for continuous systems have been presented. In this example the applicability of these methods will be investigated.

The example is concerned with design of an experiment devoted to estimate parameters from the transverse response of a 1-dimensional simply supported plane, vibrating Bernoulli-Euler beam model, see figure 6.5, subjected to the action of a transverse random load.

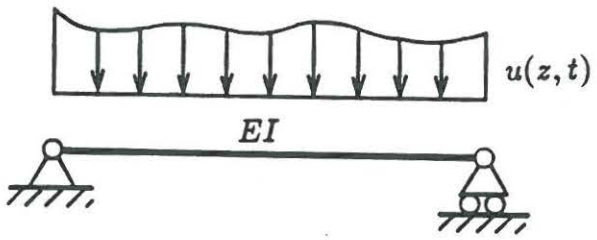

Figure 6.5: Simply supported beam subjected to random load.

It is assumed that the equation of motion for the beam is given by

$$EI\frac{\partial^4 y(z,t)}{\partial z^4} + C_d\frac{\partial y(z,t)}{\partial t} + \rho_m\frac{\partial^2 y(z,t)}{\partial t^2} = u(z,t) \tag{6.90}$$

where $y(z,t)$ is the deflection of the beam at the time t and distance z from its end. L is the beam length, ρ_m is the beam mass per unit length, C_d is the viscous

damping coefficient per unit length and EI is the bending stiffness of the beam assumed to be constant along the length of the beam. EI and ρ_m, are assumed to be known deterministic constants.

The boundary conditions describing the simply supported beam are

$$\frac{\partial^2 y(0,t)}{\partial z^2} = \frac{\partial^2 y(L,t)}{\partial z^2} = z(0,t) = z(L,t) = 0 \tag{6.91}$$

The beam load $u(z,t)$ is modelled as a zero-mean stationary Gaussian stochastic process $\{U(z,t)\}$ with a covariance given by

$$E[U(z_1,t_1)U(z_2,t_2)] = \delta(t_1 - t_2)\delta(z_1 - z_2) \tag{6.92}$$

where δ is the Dirac delta function, i.e. it is assumed that the stochastic load is white noise in both time and space with variance 1. The load is modelled as a stochastic load, due to the inevitable fact that loadings acting on structural systems are stochastic. Due to the system linearity the solution for the displacement $y(z,t)$ is as follows if a modal approach is used

$$y(z,t) = \sum_{j=1}^{\infty} q_j(t)\phi_j(z) \tag{6.93}$$

where $q_j(t)$ is a generalised coordinate and $\phi_j(z)$ is the mode shape of the jth mode. See e.g. Thomsen [22] for a solution of $q_j(t)$ and $\phi_j(z)$. The response problem is therefore in principle solved once the modal displacements are determined.

The mode shape of the jth mode is given by

$$\phi_j(z) = \sin\frac{j\pi z}{L} \tag{6.94}$$

The mode shapes satisfy the following orthogonality relations

$$\rho_m \int_0^L \phi_i(z)\phi_j(z)dz = 0 \quad \forall : i \neq j \tag{6.95}$$

$$\rho_m \int_0^L \phi_i(z)\phi_j(z)dz = M_j = \rho_m \frac{L}{2} \quad \forall : i = j \tag{6.96}$$

M_j is the generalized modal mass. $q_j(t)$ is the solution of the following second order differential equation

$$\ddot{q}_j(t) + 2\zeta_j\omega_j\dot{q}_j(t) + \omega^2 q_j(t) = \frac{p_j(z,t)}{M_j} \tag{6.97}$$

where $p_j(z,t)$ is the generalized modal loads given by

$$p_j(z,t) = \int_0^L \phi_j(z)u(z,t)dz \tag{6.98}$$

ζ_j and ω_j are the modal damping and the undamped frequencies of the jth mode, respectively, given by

$$\omega_j^2 = \frac{EI}{\rho_m}(\frac{j\pi}{L})^4 \tag{6.99}$$

$$\zeta_j = \frac{1}{2}\frac{C_d}{\rho_m \omega_j} = \frac{1}{2}\sqrt{\frac{C_d^2}{\rho_m EI}}(\frac{L}{j\pi})^2 \tag{6.100}$$

The beam is assumed to be modelled so that the lowest undamped eigenfrequency $\omega_1 = 2.0\ \pi rad/secimplyingthat\zeta_1 = 0.04$.

6.4.1 Sensor Positioning by Using the Fisher Information Matrix

In this section the applicability of the Fisher information matrix to determine the optimal locations of N_s sensors is investigated. The parameters assumed to be estimated are the modal parameters

$$\overline{\theta}^T = \{\zeta_1, \omega_1, \zeta_2, \omega_2, .., \zeta_n, \omega_n\} \tag{6.101}$$

First, the optimal locations of two sensors are determined by the Fisher information matrix as proposed in section 6.3.1, i.e the measurements are assumed to be modelled as independent random variables. Next, the optimal locations are determined directly from the definition of the Fisher information matrix.

The Fisher information matrix $\overline{\overline{J}}$, in continuous time, associated with identification of the parameters in the parameter vector $\overline{\theta}$ associated with a system described by a partial differential equation is in (6.78) given as follows for a measuring time T_f

$$J^{ij} = \sum_{k=1}^{N_s} \int_0^{T_f} \frac{\partial y(z_k,t)}{\partial \theta_i} \frac{\partial y(z_k,t)}{\partial \theta_j} \lambda_\varepsilon^{-1} dt \tag{6.102}$$

Each element of J^{ij} represents the cross-sensitivity of a measurement with respect to the response $y(z_k,t)$ at the location z_k and it is assumed that the noise is a stationary Gaussian white noise process, i.e. the measurements are assumed to be independent both in time and space coordinates.

If it is assumed that a D-optimum experiment design is selected it is shown in section 6.1.4 that the optimal locations of the N_s sensors can be found from the mode shape information if the system is lightly damped.

Since it is assumed above that the system is lightly damped the optimal location of N_s sensors can be determined by maximizing the determinant of the Fisher information matrix which is similar to

$$\max_{\overline{z}} \det \overline{\overline{J}} \sim \max_{\overline{z}} \left(\prod_{i=1}^{n} \sum_{j=1}^{N_s} \frac{1}{\lambda_\varepsilon} (\phi_i(z_j)^2) \right) \tag{6.103}$$

(6.103) follows from (6.62) which is established for a discrete system. It is assumed that the variance of the noise λ_ε is equal in each measuring point.

If $N_s = 2$ and only the modal parameters of the first two modes are of interest the determinant of the Fisher information matrix as a function of the locations of the two measuring points is shown in figure 6.6. It may be noticed that a minimum of the determinant of the parameter covariance matrix corresponds to a maximum of the determinant of the Fisher information matrix.

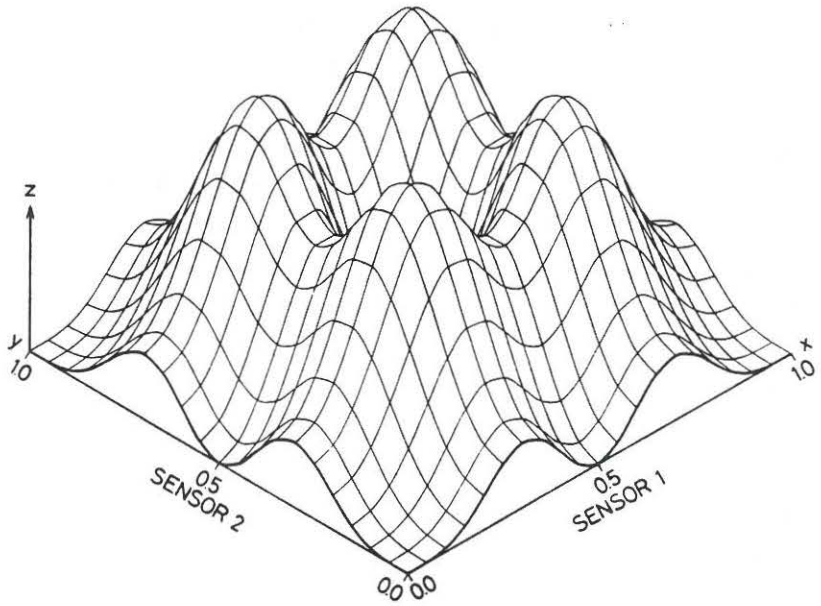

Figure 6.6: The determinant of the Fisher information matrix shown as a function of the locations of two measuring points.

It is seen from figure 6.6 that the same information will be obtained from the measurements if the sensors are placed at the same point or with the one sensor at this point and the other placed at the symmetric point.

This result implies that following question can be asked

- Is it correct that the optimal locations of the two sensors are at the same point ?

In the following this question will be investigated closer by calculating the elements in the Fisher information matrix directly from the definition and not from (6.102).

If it assumed that N number of data included in the vector $\overline{Y}$ are obtained the Fisher information matrix is given by

$$\overline{\overline{J}} = E_{\overline{Y}|\overline{\theta}_0}\left[\left(\frac{\partial \log L(\overline{\theta}, \lambda_\varepsilon)}{\partial \overline{\theta}}\right)\left(\frac{\partial \log L(\overline{\theta}, \lambda_\varepsilon)}{\partial \overline{\theta}}\right)^T\right] \tag{6.104}$$

$L(\overline{\theta}, \lambda_\varepsilon)$ is the likelihood function $L(\overline{\theta}, \lambda_\varepsilon) = f_{\overline{Y}}(\overline{y}|\overline{\theta})$ where $f_{\overline{Y}}(\overline{y}|\overline{\theta})$ is the condi-

tional N-dimensional joint probability density function.

(6.104) can be written

$$\overline{\overline{J}} = -E_{\overline{Y}|\overline{\theta}_0}\left[\frac{\partial^2 \log f_{\overline{Y}}(\overline{y}|\overline{\theta}))}{\partial \overline{\theta}^2}\right] \tag{6.105}$$

By taking the expectation (6.105) becomes

$$\overline{\overline{J}} = -\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{\partial^2 \log f_{\overline{Y}}(\overline{y}|\overline{\theta}))}{\partial \overline{\theta}^2} f_{\overline{Y}}(\overline{y}|\overline{\theta})d\overline{y} \tag{6.106}$$

It is seen from (6.105) that a $N \times N_s$-dimensional integral has to be solved if the Fisher information matrix is to be estimated directly from the definition which is much more cumbersome than using (6.102).

To see the difference between calculating the Fisher information matrix directly from the definition and from (6.102) it is assumed that an observation is taken only once at each measuring point simultaneously. Assuming two measuring points the integral in (6.106) becomes 2-dimensional.

Since the response is assumed to be Gaussian with zero mean the 2-dimensional probability density function for the two measuring points can now be written by a 2-dimensional Gaussian joint density function of the two continuous random variables Y_1 and Y_2

$$f_{Y_1,Y_2}(y_1,y_2|\overline{\theta}) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}^2}}\exp\left(-\frac{1}{2(1-\rho_{12}^2)}((\frac{y_1}{\sigma_1})^2 - 2\rho_{12}\frac{y_1y_2}{\sigma_1\sigma_2} + (\frac{y_2}{\sigma_2})^2)\right) \tag{6.107}$$

where σ_1 and σ_2 are the standard deviations of the response at the two measuring points, respectively. ρ_{12} is the correlation coefficient of Y_1 and Y_2

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2} \tag{6.108}$$

where σ_{12} is the covariance between Y_1 and Y_2

The two random variables are given by

$$Y_1 = Y(z_1,t_1) \tag{6.109}$$

$$Y_2 = Y(z_2,t_2) \tag{6.110}$$

where $\{Y(z,t)\}$ is the stochastic response process.

The logarithm of the joint density function is

$$\log f_{Y_1,Y_2}(y_1,y_2|\overline{\theta}) = -\log 2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}^2} - \frac{1}{2(1-\rho_{12}^2)}\left((\frac{y_1}{\sigma_1})^2 - 2\rho_{12}\frac{y_1y_2}{\sigma_1\sigma_2} + (\frac{y_2}{\sigma_2})^2\right) \tag{6.111}$$

Now the first term in (6.111) is denoted a_1 and the second term a_2

$$a_1 = -\log 2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}^2} \tag{6.112}$$

$$a_2 = -\left(\frac{1}{2(1-\rho_{12}^2)}((\frac{y_1}{\sigma_1})^2 - 2\rho_{12}\frac{y_1y_2}{\sigma_1\sigma_2} + (\frac{y_2}{\sigma_2})^2)\right) \tag{6.113}$$

The second derivative in (6.106) can now be written

$$\frac{\partial^2 \log f_{\overline{Y}}(\overline{y}|\overline{\theta})}{\partial\overline{\theta}^2} = \frac{\partial^2 a_1}{\partial\overline{\theta}^2} + \frac{\partial^2 a_2}{\partial\overline{\theta}^2} \tag{6.114}$$

In order to write down expressions for the expectation of the two terms in (6.114) the following auxiliary terms are introduced.

$$a_3 = \sigma_1^2\sigma_2^2 - \sigma_{12}^2 \tag{6.115}$$

$$a_4 = \frac{\partial(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)}{\partial\theta_i} = (\frac{\partial\sigma_1^2}{\partial\theta_i}\sigma_2^2 + \frac{\partial\sigma_2^2}{\partial\theta_i}\sigma_1^2 - \frac{\partial\sigma_{12}^2}{\partial\theta_i}) \tag{6.116}$$

$$a_5 = \frac{\partial(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)}{\partial\theta_j} = (\frac{\partial\sigma_1^2}{\partial\theta_j}\sigma_2^2 + \frac{\partial\sigma_2^2}{\partial\theta_j}\sigma_1^2 - \frac{\partial\sigma_{12}^2}{\partial\theta_j}) \tag{6.117}$$

$$\begin{aligned} a_6 = \frac{\partial^2(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)}{\partial\theta_i\partial\theta_j} =& (\frac{\partial^2\sigma_1^2}{\partial\theta_i\partial\theta_j}\sigma_2^2 + \frac{\partial\sigma_2^2}{\partial\theta_j}\frac{\partial\sigma_1^2}{\partial\theta_i} + \frac{\partial^2\sigma_2^2}{\partial\theta_i\partial\theta_j}\sigma_1^2 \\ &+ \frac{\partial\sigma_2^2}{\partial\theta_i}\frac{\partial\sigma_1^2}{\partial\theta_j} - \frac{\partial^2\sigma_{12}^2}{\partial\theta_i\partial\theta_j}) \end{aligned} \tag{6.118}$$

$$a_7 = \frac{\partial^2\sigma_1^2}{\partial\theta_i\partial\theta_j} \tag{6.119}$$

$$a_8 = \frac{\partial^2\sigma_2^2}{\partial\theta_i\partial\theta_j} \tag{6.120}$$

$$a_9 = \frac{\partial^2\sigma_{12}}{\partial\theta_i\partial\theta_j} \tag{6.121}$$

$$a_{10} = \frac{\partial\sigma_1^2}{\partial\theta_i} \tag{6.122}$$

$$a_{11} = \frac{\partial\sigma_2^2}{\partial\theta_i} \tag{6.123}$$

$$a_{12} = \frac{\partial\sigma_{12}}{\partial\theta_i} \tag{6.124}$$

$$a_{13} = \frac{\partial\sigma_1^2}{\partial\theta_j} \tag{6.125}$$

$$a_{14} = \frac{\partial \sigma_2^2}{\partial \theta_j} \tag{6.126}$$

$$a_{15} = \frac{\partial \sigma_{12}}{\partial \theta_j} \tag{6.127}$$

Now, the expectation of the second derivative of a_1 with respect to $\overline{\theta}$ becomes

$$E_{Y|\overline{\theta}_0}\left[\frac{\partial^2 a_1}{\partial \theta_i \partial \theta_j}\right] = \left(\frac{a_6 a_3 - a_4 a_5}{2a_3^2}\right) \tag{6.128}$$

The expectation of the second derivative of a_2 with respect to $\overline{\theta}$ becomes

$$\begin{aligned} E_{Y|\overline{\theta}_0}\left[\frac{\partial^2 a_2}{\partial \theta_i \partial_j}\right] = & \frac{\frac{\sigma_1^2}{2}(a_8 a_3 - a_{11} a_5) + \frac{\sigma_2^2}{2}(a_7 a_3 - a_{10} a_5) - \sigma_{12}(a_9 a_3 - a_{12} a_5)}{a_3^2} + \\ & \left(\frac{\frac{\sigma_1^2}{2}((a_6 \sigma_2^2 + a_4 a_{14}) a_3^2 - 2a_4 a_3 a_5 a_{14})}{a_3^4} + \right. \\ & \frac{\frac{\sigma_2^2}{2}((a_6 \sigma_1^2 + a_4 a_{13}) a_3^2 - 2a_4 a_3 a_5 a_{13})}{a_3^4} + \\ & \left. \frac{\frac{\sigma_{12}}{2}((a_6 \sigma_{12} + a_4 a_{15}) a_3^2 - 2a_4 a_3 a_5 a_{15})}{a_3^4} \right) \end{aligned} \tag{6.129}$$

It is seen from (6.115)-(6.129) that the gradients of the variances and covariance of Y_1,Y_2 have to be determined.

The variance and covariance of $Y_1(z_1,t)$,$Y_2(z_2,t)$ are given by the cross-correlation function $R_{YY}(z_l,t_1;z_2,t_2)$ of the response in the two points z_1 and z_2 defined by

$$R_{YY}(z_1,t_1;z_2,t_2) = E[Y(z_1,t_1), Y(z_2,t_2)] \tag{6.130}$$

Since the response is assumed to be zero mean (6.130) implies the following, if $t_1 = t_2$

$$\sigma_1^2 = E[(Y(z_1,t_1) - E[Y(z_1,t_1)])(Y(z_1,t_1) - E[Y(z_1,t_1)])] = R_{YY}(z_1,z_1;0) \tag{6.131}$$

$$\sigma_2^2 = E[(Y(z_2,t_2) - E[Y(z_2,t_2)])(Y(z_2,t_2) - E[Y(z_2,t_2)])] = R_{YY}(z_2,z_2;0) \tag{6.132}$$

$$\sigma_{12} = E[(Y(z_1,t_1) - E[Y(z_1,t_1)])(Y(z_2,t_2) - E[Y(z_2,t_2)])] = R_{YY}(z_1,z_2;0) \tag{6.133}$$

The cross-correlation function of the stationary response follows from

$$R_{YY}(z_1,t_1;z_2,t_2) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \phi_i(z_1)\phi_j(z_2) R_{Q_i Q_j}(t_1,t_2) \tag{6.134}$$

where the cross-correlation function $R_{Q_iQ_j}(t_1,t_2)$ of the stationary stochastic $\{Q(t)\}$ modal response follows from

$$R_{Q_iQ_j}(t_1,t_2) = \int_0^{t_1}\int_0^{t_2} h_i(t_1-\tau_1)h_j(t_2-\tau_2)R_{P_iP_j}(t_1,t_2)d\tau_1 d\tau_2 \qquad (6.135)$$

where the cross-correlation function $R_{P_iP_j}(0)$ of the stationary stochastic $\{P(t)\}$ modal load follows from

$$R_{P_iP_j}(t_1,t_2) = \int_0^{L}\int_0^{L} \phi_i(u_1)\phi_j(u_2)R_{UU}(u_1,t_1;u_2,t_2)du_1 du_2 \qquad (6.136)$$

Since the cross-correlation function $R_{UU}(u_1,t_1;u_2,t_2)$ of the load process is given by (6.92) the cross-correlation in (6.136) is given by

$$R_{P_iP_j}(t_1,t_2) = \int_0^{L}\int_0^{L} \phi_i(u_1)\phi_j(u_2)\delta(t_1-t_2)\delta(u_1-u_2)du_1 du_2 \qquad (6.137)$$

(6.137) implies that the cross-correlation function in (6.130) can be written

$$R_{YY}(z_1,z_2;0) = \sum_{i=1}^{\infty}\phi_i(z_1)\phi_i(z_2)\frac{\pi}{2\omega_i^3\zeta_i} \qquad (6.138)$$

Now, it is easy to calculate the derivatives in (6.114) and thus the Fisher information matrix in (6.105).

In figure 6.7 the determinant of the Fisher information matrix is shown for different locations of the two sensors. It is assumed that only the modal parameters of the first two modes are of interest. The points in figure 6.7 where $z_1 = z_2$ are determined by using a 1-dimensional probability density function in (6.106) instead of the 2-dimensional function. It may be noticed that the Fisher information matrix is not defined for $z_1 = z_2 = 0 = L$. The points $z_i = 0$ for $z_j \neq 0, L$ are determined in the same way as the points $z_i = z_j$.

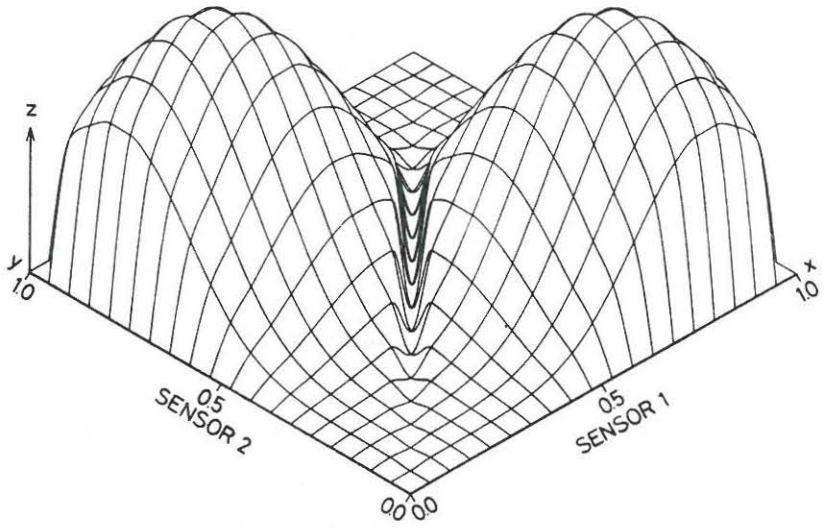

Figure 6.7: The determinant of the Fisher information matrix shown as a function of the location of two sensors.

By investigating figure 6.7 it is seen that it is not optimal to place two sensors at the same measuring point. This disagrees with the result shown in figure 6.6. The disagreement is due to the assumption that the measurements can be considered as statistically independent random variables. This implies that the spatial correlation is not taken into account in the calculations of the results shown in figure 6.6. The results shown in figure 6.7 are obtained from calculations where the spatial correlation is taken into account.

The optimal locations of the two sensors are $z_1 = 0.242L$ and $z_2 = 0.758L$. Intuitively, it also seems to be more correct to get information from two different measuring points instead of information from one measuring point. If two sensors are placed at the same point and it is assumed that the observations are not encumbered with random measurement noise, the sensors will measure the same realization of the stochastic response process. Two identical outcomes of a stochastic process do not give more information than one. It may be noticed that two realizations of a stochastic process obtained from the same measuring point are identical if they are measured simultaneously and if it is assumed that they are not encumbered with random measuring noise. It is also seen from the above that it is much simpler to calculate the Fisher information matrix in the case where it can be assumed that the measurements are statistically independent. However, as it is shown above, it is not a good approach when a continuous mechanical system subjected to a random load is considered.

If the measurements are encumbered with noise it can be interesting to look into the optimal locations of the sensors are sensitive to the variance of the random noise.

It is assumed that the measurements are given by

$$y^m(z,t) = y(z,t) + e(z,t) \tag{6.139}$$

where the realization $e(z,t)$ of the noise process $\{\mathcal{E}(z,t)\}$ only models the random measuring noise. In figure 6.8 the determinant of the Fisher information matrix is shown for different locations of the two sensors, i.e. it is the same figure as figure 6.7 except that it is assumed that the noise-to-signal ratio γ is 0.2 in figure 6.8. The noise-to-signal ratio γ is given by

$$\gamma = \sqrt{\frac{\lambda_\mathcal{E}}{\sigma^2}} \tag{6.140}$$

σ^2 is the variance of the response for $z = L/2$

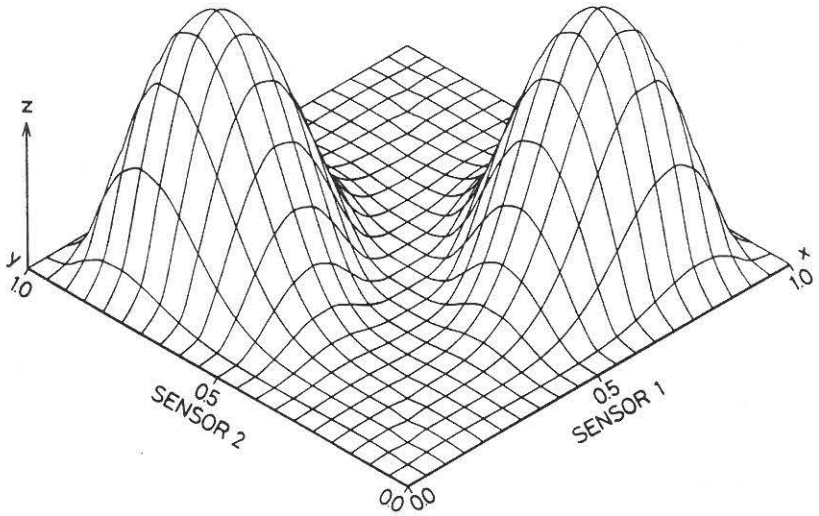

Figure 6.8: The determinant of the Fisher information matrix shown as a function of the locations of two sensors. The noise-to-signal ratio is assumed to be 0.2.

Comparing figure 6.7 and figure 6.8 it is seen, as expected, that the amount of information from the measurements obtained from the two measuring points decreases when the measurements are encumbered with noise. It is clearly seen for measuring points placed near the ends of the beam and for the points placed near the middle of the beam.

In figure 6.9 the optimal locations of the two sensors are shown as functions of the noise-to-signal ratio. The full line in figure 6.9 shows results where it is assumed that it is the modal parameters of the first two modes which are of interest. The dashed line shows results where it is assumed that it is the modal parameters (ζ_1, ω_1) of the first mode which are of interest, i.e. the Fisher information matrix is a 2×2 matrix.

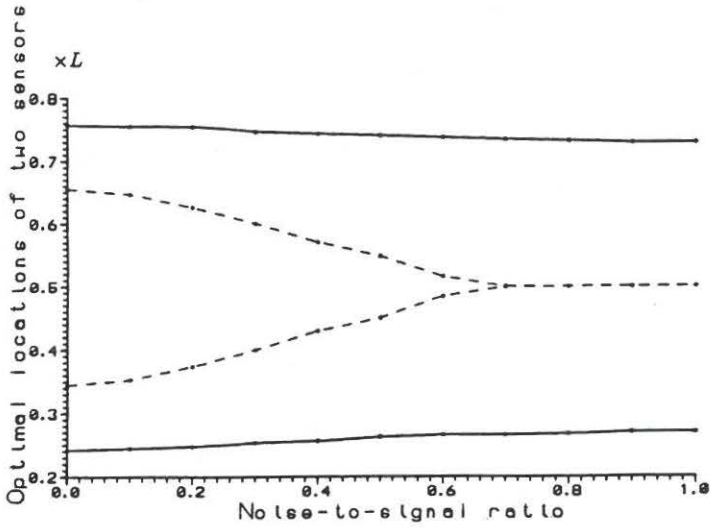

Figure 6.9: Optimal locations of two sensors as function of the noise-to-signal ratio γ.

It is seen from figure 6.9 that the optimal locations of the two measuring points are sensitive to the variance of the noise. It is seen that the optimal locations are more sensitive to the variance of the noise when it is the parameters of the first mode which are of interest. Then the optimal locations of the sensors are going against, as expected, the optimal location of one measuring point, $z = L/2$. From this result it could be expected for an increasing number of sensors that the optimal locations of the sensors become less sensitive to the variation of the noise-to-signal ratio.

Figure 6.10 shows the relative change in the determinant of the Fisher information matrix, corresponding to the optimal locations of the sensors, as functions of the noise-to-signal ratio γ. The dashed line and the full line, respectively, correspond to the lines in figure 6.9.

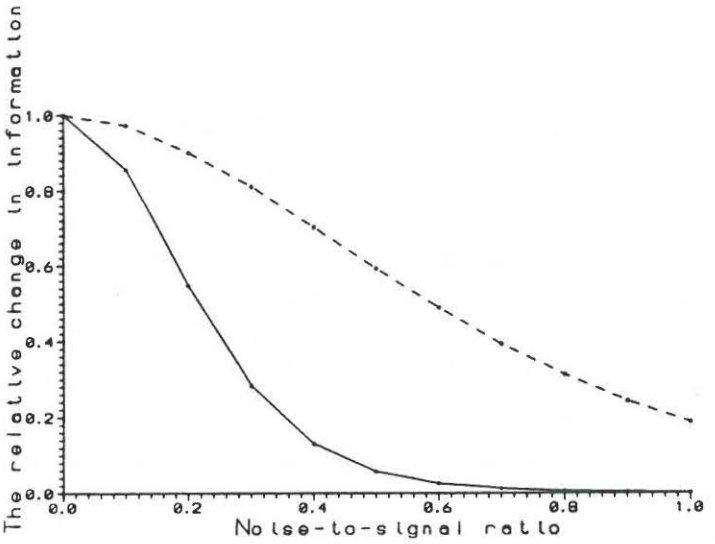

Figure 6.10: The change in the determinant of the Fisher information matrix as a function of the noise-to-signal ratio γ.

It is seen from figure 6.10 that the relative loss of information for increasing variance of the noise is larger when the parameters of the first two modes are of interest than when it is the parameters of the first mode which are of interest.

Figure 6.11 shows, for increasing noise-to-signal ratio, the ratio κ_1

$$\kappa_1 = \frac{\det \overline{\overline{J}}_1}{\det \overline{\overline{J}}_2} \tag{6.141}$$

where $\det \overline{\overline{J}}_1$ is the determinant of the Fisher information matrix for the two measuring points placed at the optimal points assuming noiseless measurements and $\det \overline{\overline{J}}_2$ is the determinant of the Fisher information matrix for optimally located sensors, i.e. the ratio shows the loss in information by placing the sensors without taking into account that the measurements are encumbered with noise.

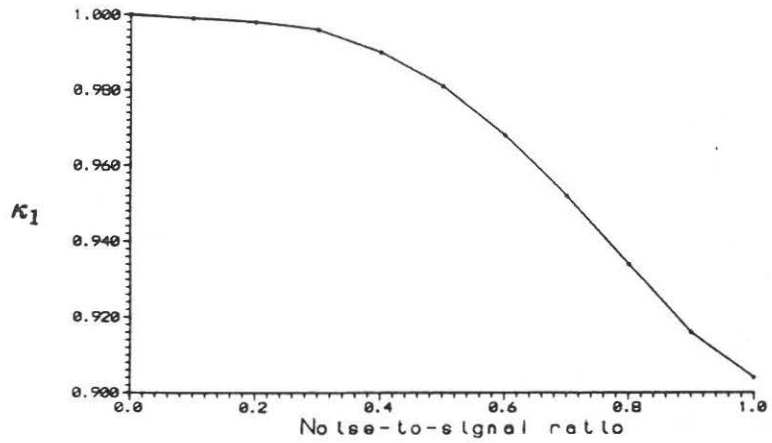

Figure 6.11: The ratio κ_1 shown as a function of the noise-to-signal ratio γ.

It is seen from figure 6.11 that only a little loss of information is obtained if the sensors are placed without taking into account that the measurements are encumbered with noise. This is an important result since the noise-to-signal ratio is normally not available prior to the experiment.

Figure 6.12 shows the ratio κ_2 for increasing noise-to-signal ratio γ,

$$\kappa_2 = \frac{\det \overline{\overline{J}}_4}{\det \overline{\overline{J}}_3} \tag{6.142}$$

where $\det \overline{\overline{J}}_3$ is the amount of information obtained if two sensors are optimally placed and $\det \overline{\overline{J}}_4$ is the amount of information obtained if one sensor is optimally placed.

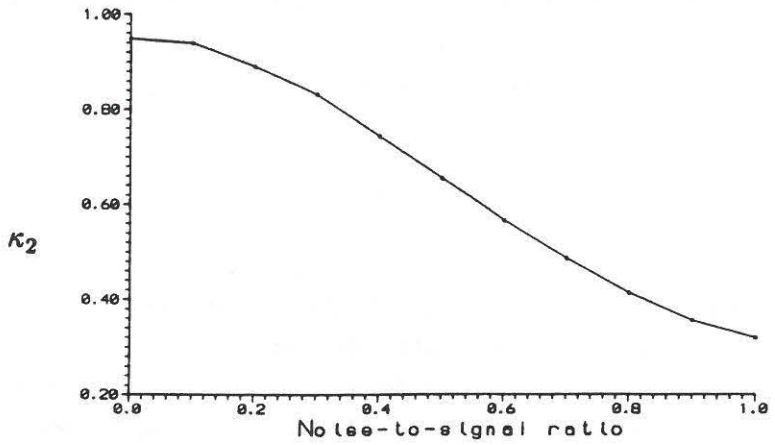

Figure 6.12: The ratio κ_2 shown as a function of the noise-to-signal ratio γ. .

Figure 6.12 shows, for noiseless measurements, that the amount of information obtained using one sensor is almost equal to the amount of information obtained using two sensors. It is assumed that it is the modal parameters of the first mode which are of interest. Further, it is seen for increasing noise-to-signal ratio that the ratio κ_2 becomes smaller. This means that the amount of information obtained with two sensors becomes larger compared with the amount of information obtained with one sensor.

This result raises the question:

- When shall additional sensors be used ?

In order to answer this question it is necessary to take into account the cost of using an additional sensor. Further, it is also necessary to consider the increase in information value by using an additional sensor. This problem will be considered later in this thesis.

The results in figures 6.10-6.12 indicate that the optimal locations of the sensors are not sensitive to the noise-to-signal ratio if the aim of the experiment is to determine the modal parameters of more than one mode. On the other hand it is seen from figure 6.10-6.12 that good prior information about the noise-to-signal ratio is necessary if the aim is to compare the information which can be obtained using different number of sensors.

6.4.2 Sensor Positioning by Using the Concept of Entropy

In this section the optimal locations of N_s sensors are determined by using the concept of entropy. It may be noticed that by using the concept of entropy it is not possible to take into account the kind of parameters to be estimated from the

experiment. Thus, the same optimal locations of the sensors are found if the aim of the experiment is to get information about the modal parameters or of other parameters. However, it will be seen in the following sections that a design based on the concept of entropy under some circumstances will coincide with a design obtained from a criterion taking kind of parameters into account.

From section 6.3.2 the entropy of the measurements can be written

$$H_{\overline{Y}}(\overline{y}) = \log \sqrt{(2\pi e)^{N_s} |\overline{\overline{R}}|} \tag{6.143}$$

It is seen that for Gaussian random variables the entropy is maximized where the determinant of the correlation matrix $\overline{\overline{R}}$ attains its maximum. Therefore if two measuring points are considered the optimality criterion is given by

$$\max_{\overline{z}} H_{\overline{Y}}(\overline{y}) \sim \max_{\overline{z}} \Big(R_{YY}(z_1, z_1; 0) R_{YY}(z_2, z_2; 0) - R_{YY}(z_1, z_2; 0)^2 \Big) \tag{6.144}$$

In figure 6.13 the entropy is shown as a function of the optimal locations of two sensors.

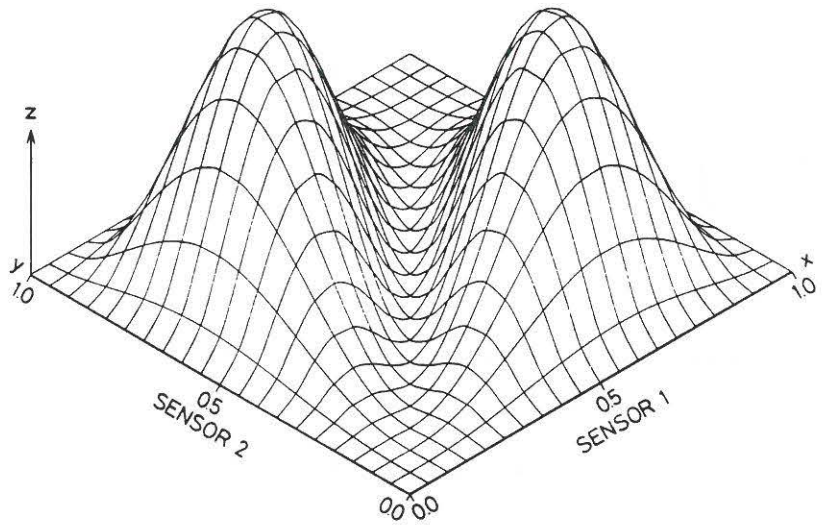

Figure 6.13: The entropy shown as a function of the locations of the two measuring points.

It is seen from figure 6.13 that the concept of entropy implies that it is not optimal to place the two sensors at the same measuring point, i.e. that the concept of

entropy criterion takes the spatial correlation into account. The optimal locations of the sensors for different number of sensors N_s are shown in table 6.1.

N_s	z_1	z_2	z_3	z_4
1	0.500L			
2	0.311L	0.689L		
3	0.226L	0.500L	0.774L	
4	0.230L	0.442L	0.558L	0.770L

Table 6.1: Optimal locations of different number of sensors. L is the beam length.

Figure 6.14 shows, for increasing noise-to-signal ratio γ, the ratio κ_3

$$\kappa_3 = \frac{\det \overline{\overline{J}}_5}{\det \overline{\overline{J}}_6} \tag{6.145}$$

where $\det \overline{\overline{J}}_5$ is the determinant of the Fisher information matrix if the two measuring points are placed according to the concept of entropy and $\det \overline{\overline{J}}_6$ is the determinant of the Fisher information matrix if the two sensors are placed according to the Fisher information matrix.

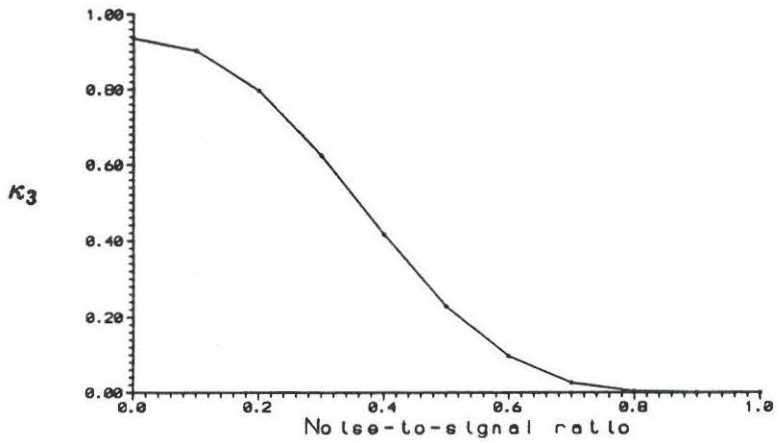

Figure 6.14: The ratio κ_3 shown as a function of the noise-to-signal ratio γ.

It is seen from figure 6.14 that the reduction in the amount of information obtained by placing the two sensors by the concept of entropy is increasing for increasing noise-to-signal ratio. For small noise-to-signal ratios, it is seen that the concept of entropy criterion gives results which are close to those from the Fisher information criterion. Thus, one could conclude that the concept of entropy criterion may be proper for the design of identification experiments. However, as mentioned before,

the concept of entropy criterion does not depend on the kind of parameters to be estimated from the experiment. This implies that the optimal locations of two sensors, obtained by the concept of entropy criterion, would be the same if the aim of the experiment is to get information about the modal parameters of the first mode or the first three modes, etc..

In the following example it will be shown that the concept of entropy criterion measures something which is not connected with the aim of experiment design for the parametric identification.

6.4.3 Sensor Positioning by Using the Best Linear Unbiased Estimator

In this section an example corresponding to an example in Kazimierczyk [31] and Kazimierczyk [32] is given to illustrate some important results. E.g. the example gives some insight into the influence of asymmetry of loading on the optimal locations of sensors and the sensitivity of the error caused by using a suboptimal design on the number of sensors. Further, the example is important since it presents an experiment design method which can take into account that measurements in general are statistically independent.

It is assumed that the beam model is now subjected to a load modelled by a stationary Gaussian stochastic process $\{U(z,t)\}$ with a covariance given by

$$E[U(z_1,t_1),U(z_2,t_2)] = \delta(t_1-t_2)\delta(z_1-z_2) \tag{6.146}$$

and a mean given by

$$E[U(z,t)] = V\eta_0 z^{V-1} \tag{6.147}$$

where V will be referred to as the mean loading mode.

On these assumptions, due to the linearity of the model, the solution $y(z)$ can be represented as

$$y(z) = y_0(z) + y_1(z) \tag{6.148}$$

where $y_0(z)$ is the solution of (6.90) with the right side equal to the mean load, and $y_1(z)$ is the solution of (6.90) with the right side equal to $u(z,t) - E[U(z,t)]$. $y(z)^0$ is the static deflection of the beam which can be easily determined from the uniform boundary conditions (6.91) and from equation (6.90) which here takes the simple form

$$EI\frac{d^4}{dz^4}y_0(z) = E[U(z,t)] \tag{6.149}$$

Hence, for $V = 1$ the solution is

$$y_0(z) = \frac{\eta_0}{24EI}(z^4 - 2Lz^3 + L^3z) \tag{6.150}$$

It is seen that (6.150) is linear in the parameter η_0. Thus, if the aim of the experiment is to determine η_0 the method proposed in section 6.3.3 can be used to determine the optimal locations of N_s number of sensors.

To establish the covariance matrix of the parameter estimate in (6.89) the correlation matrix $\overline{\overline{R}}$ of the response has to be determined . An element of $\overline{\overline{R}}$ is given by (6.138).

Using (6.89) and (6.138) the covariance matrix of the parameter η_0 can now be written down. For two measuring points and for the first mode of mean loading ($V = 1$) the covariance takes the form

$$\begin{aligned}\overline{\overline{C}}_{\hat{\theta}_N} = &\left(\frac{24EI}{\eta_0}\right)^2 \left(R_{YY}(z_1,z_1;0)R_{YY}(z_2,z_2;0) - R_{YY}(z_1,z_2;0)^2\right) \quad / \\ &\Big((z_1^4 - 2Lz_1^3 + L^3z_1)^2 R_{YY}(z_2,z_2;0) - \\ &2(z_1^4 - 2Lz_1^3 + L^3z_1)(z_2^4 - 2Lz_2^3 + L^3z_2)R_{YY}(z_1,z_2;0) + \\ &(z_2^4 - 2Lz_2^3 + L^3z_2)^2 R_{YY}(z_1,z_1;0)\Big) \end{aligned} \tag{6.151}$$

In figure 6.15 the inverse of the covariance matrix (6.151) as a function of the locations of two sensors is shown. It is clearly seen from (6.151) that during digital calculations the infinite series has to be substituted by the finite expression. The most natural solution is to use the finite series instead of the infinite one. The results shown in figure 6.15 are calculated by approximating the infinite series by a finite series with N_M terms. In Kazimierczyk [31] and Kazimierczyk [32] the influence of the choice of N_M is investigated. E.g it is found that for two sensors not less than four terms should be used and for three sensors not less than five terms. It may be noticed that figure 6.15 does not show the results for (z_1=z_2=0), (z_1=z_2=L) and (z_1=0,z_2=L), but results close to these points.

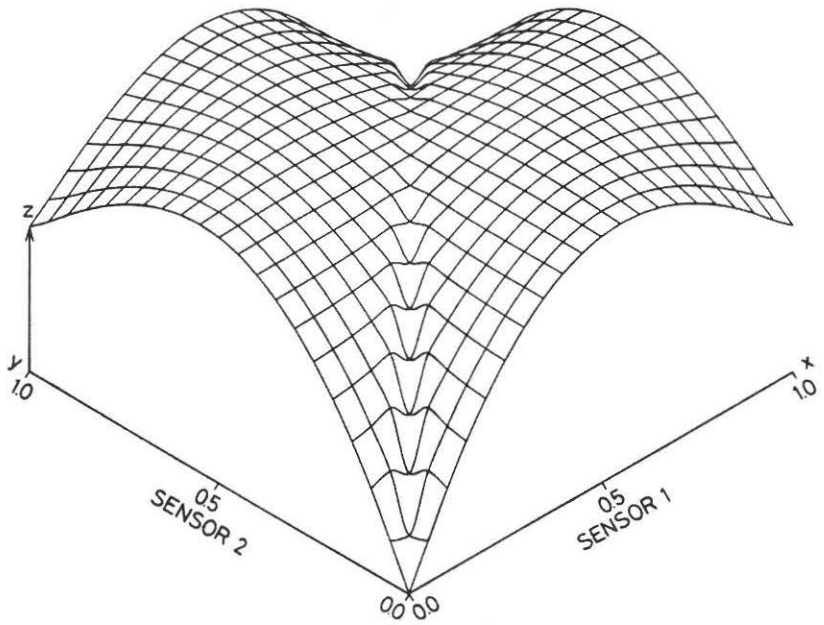

Figure 6.15 The inverse of the covariance matrix shown as a function of the locations of two sensors.

Figure 6.15 shows that it is not optimal to place two sensors at the same measuring point. Further, it is seen that the 3D-curve is very flat near its maxima. This causes difficulties in the precise choice of the optimal design on the one hand, but it also means that some imperfections in the design or in the practical positioning of the sensors results in relatively small increase of error.

The optimal locations of the sensors for different number of sensors N_s are shown in table 6.2

N_s	z_1	z_2	z_3	z_4
1	0.500L			
2	0.310L	0.690L		
3	0.223L	0.500L	0.777L	
4	0.229L	0.444L	0.556L	0.771L

Table 6.2: Locations for different number of sensors. L is the beam length.

Table 6.2 shows the optimal locations of different number of sensors. It is seen that the optimal locations are very similar with the optimal locations found by the concept of entropy criterion. Thus, one could conclude that the concept of entropy criterion may also be proper for the design of identification experiments. However, in Kazimierczyk [31] and Kazimierczyk [32] it is shown that this is not

true. In figure 6.16 it can be seen that this coincidence holds at most in the case where the mean load V is chosen to be equal to one. In figure 6.16 the optimal locations of different number of sensors N_s are shown as function of the mean load V. It may be noticed that (6.151) cannot be used if $V \neq 1$. Then (6.149) has to be solved with the right side equal to a mean load where $V \neq 1$.

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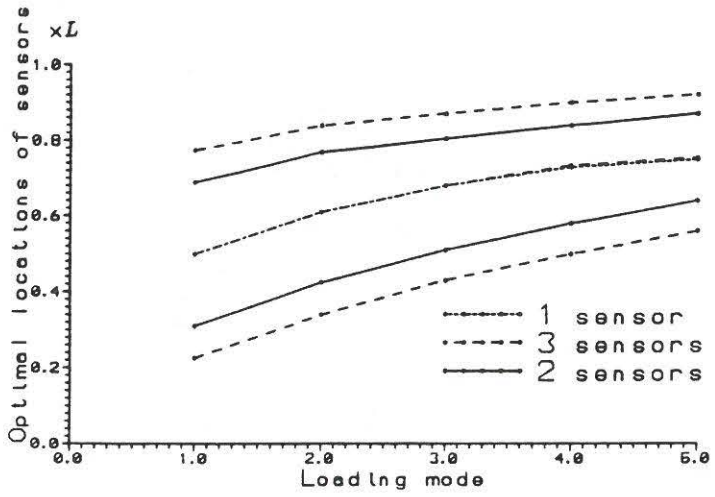

Figure 6.16: Optimal locations of sensors shown as a function of the mean loading mode V.

It is seen from figure 6.16 that the locations optimal for one sensor are identical with the positions optimal for the middle of three sensors. Such result could have been expected but it is by no means obvious. The same can be observed for the central sensor for five sensors. Further, it is seen from figure 6.16 that the concept of entropy criterion only gives the optimal experiment design for $V = 1$. The concept of entropy criterion is not affected by the change of the mode of the mean loading.

In figure 6.17 the error in the covariance caused by using locations optimal for $V = 1$ where $V > 1$ is shown.

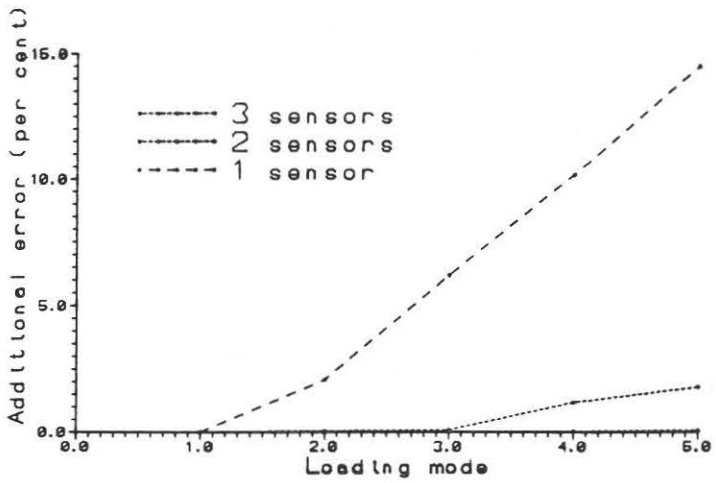

Figure 6.17: The percentage of the error in the covariance caused by using $V = 1$ where $V > 1$ shown as a function of the mean loading mode V.

It is seen from figure 6.17 that the sensitivity of optimal design on the loading mode V decreases with the increased number of sensors. This result seems to be an important result. However, it is obvious , as mentioned in section 6.4.2, that the concept of entropy criterion does not measure anything which is connected with the aim of the design of experiment even though it works properly under some conditions.

Figure 6.18 shows the change in the covariance of the parameter estimate as a function of the number of sensors.

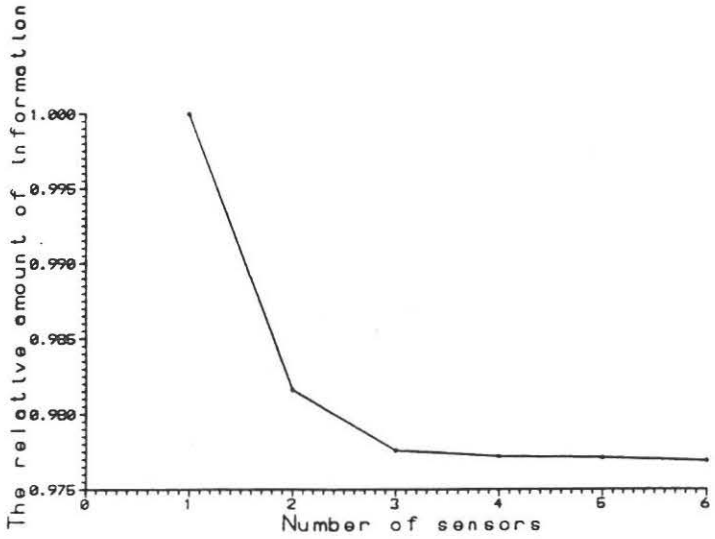

Figure 6.18: Change in the covariance of the parameter estimate as a function of the number of sensors.

It is seen from figure 6.18 that increasing the number of sensors leads to a decrease in the covariance of the parameter estimate. However, the decrease is very small for $N_s > 2$. Nevertheless, according to figure 6.18 the use of many measuring points is advisable not only to decrease the estimation error at the optimal points but also to make the design less sensitive.

6.4.4 Sensor Positioning by Using an Asymptotic Expression for the Parameter Covariance Matrix

The examples given in sections 6.4.1, 6.4.2 and 6.4.3 concerned with the validity of different methods to determine the optimal locations of N_s number of sensors for the parametric estimation of a continuous mechanical system subjected to a random load have shown that measurements can hardly be modelled by independent random variables when a civil engineering structure is subjected to a random load. Thus, in general both space and time dependence and correlation have to be taken into account. Above it is seen that this is possible if the experiment design is based on a scalar measure of the parameter covariance matrix estimated directly from the definition of the Fisher information matrix. However, in general, it is difficult/or cumbersome to evaluate the Fisher information matrix for more than a 2-dimensional problem. Further, it is seen that the method based on a BLUE estimator is available too if the parameters to be estimated can be written by a model linear in the parameters.

However, due to the inevitable fact that the responses in different locations of a civil engineering structure are correlated, through the mode shapes, and that the loadings acting on real civil engineering structures are random, leads to the necessity of the use of a method for design of an experiment for parametric estimation which take into account

- that measurements in different measuring points are dependent, and
- the method must be able to estimate the parameter covariance matrix if the parameters to be estimated are modelled by a model non-linear in the parameters.

Here, it will be proposed to use following expression, see Ljung [33], to estimate the parameter covariance matrix in the general problem where the measurements are obtained from a multi-output system subjected to random load

$$\overline{\overline{P}}_{\hat{\theta}_N} = E[\overline{\overline{\Psi}}(t,\overline{\theta}_0)\overline{\overline{\Xi}}\,\overline{\overline{\Psi}}^T(t,\overline{\theta}_0)]^{-1}E[\overline{\overline{\Psi}}(t,\overline{\theta}_0)\overline{\overline{\Omega}}\,\overline{\overline{\Psi}}^T(t,\overline{\theta}_0)]E[\overline{\overline{\Psi}}(t,\overline{\theta}_0)\overline{\overline{\Xi}}\,\overline{\overline{\Psi}}^T(t,\overline{\theta}_0)]^{-1} \tag{6.152}$$

The covariance of the parameter estimates is then given by

$$\overline{\overline{C}}_{\hat{\theta}_N} \approx \frac{1}{N}\overline{\overline{P}}_{\hat{\theta}_N} \tag{6.153}$$

(6.152) is the expression to estimate the parameter covariance matrix for parametric estimation of a system when the response is measured at more than one point.

(6.152) corresponds to the expression (2.22) in chapter 2 for the 1- dimensional problem.

A realization $\psi(t,\bar{\theta}_0)_{ij}$ of the stochastic process $\{\Psi(t,\bar{\theta}_0)_{ij}\}$ is defined by

$$\psi(t,\bar{\theta}_0)_{ij} = -\frac{\partial \epsilon_j(t,\theta_0)}{\partial \theta_i} \tag{6.153}$$

$\{\overline{\overline{\Psi}}(t,\bar{\theta}_0)\}$ is a matrix of dimension $d \times p$ where d and p are the dimensions of the $\bar{\theta}$ and $\bar{\epsilon}$ vectors, respectively.

$\overline{\overline{\Omega}}$ and $\overline{\overline{\Xi}}$ are $p \times p$ matrices.

If a quadratic criterion is used to minimize the prediction errors the sum of errors V_N is given by

$$V_N = \sum_{i=1}^{N} \bar{\epsilon}^T(t_i,\bar{\theta}_0)\overline{\overline{W}}^{-1}\bar{\epsilon}(t_i,\bar{\theta}_0) \tag{6.154}$$

where $\overline{\overline{W}}$ is a weighting matrix.

(6.154) implies that

$$\overline{\overline{\Xi}} = \overline{\overline{W}}^{-1} \tag{6.155}$$

and

$$\overline{\overline{\Omega}} = \overline{\overline{W}}^{-1}\overline{\overline{\Lambda}}_0\overline{\overline{W}}^{-1} \tag{6.156}$$

where $\overline{\overline{\Lambda_0}}$ is given by

$$\overline{\overline{\Lambda}}_0 = E[\bar{e}_0(t)\bar{e}_0^T(t)] \tag{6.157}$$

$\bar{\epsilon}(t,\bar{\theta}_0) = \bar{e}_0(t)$ where $\bar{e}_0(t)$ is a sequence of independent zero mean vectors.

It is seen that using (6.152) to estimate the parameter covariance matrix in the multivariable case introduces a new issue

- How should the different components of the vector $\bar{\epsilon}(t,\bar{\theta})$ be weighted in relation to one another?

It is straightforward, see Ljung [33], to establish that

$$\overline{\overline{P}}_{\hat{\bar{\theta}}_N}(\overline{\overline{W}}) \geq \overline{\overline{P}}_{\hat{\bar{\theta}}_N}(\overline{\overline{\Lambda}}_0) \quad \forall : \overline{\overline{W}} \tag{6.158}$$

where $\overline{\overline{W}}$ is a symmetrically positive definite weighting matrix. $\overline{\overline{P}}_{\hat{\bar{\theta}}_N}(\overline{\overline{W}})$ shows that the covariance matrix is estimated by using $\overline{\overline{W}}$ as a weighting matrix in (6.154). It is seen from (6.158) that the best weighting matrix is

$$\overline{\overline{W}} = \overline{\overline{\Lambda}}_0 \tag{6.159}$$

If the choice of weighting matrix $\overline{\overline{W}}$ in the criterion (6.154) is $\overline{\overline{W}} = \overline{\overline{\Lambda}}_0$ the parameter covariance matrix is given by

$$\overline{\overline{P}}_{\hat{\bar{\theta}}_N} = E[\overline{\overline{\Psi}}(t,\bar{\theta}_0)\overline{\overline{\Lambda}}_0^{-1}\overline{\overline{\Psi}}^T(t,\bar{\theta}_0)]^{-1} \tag{6.160}$$

By using (6.160) it is assumed that the prediction errors $\overline{\epsilon}(t, \overline{\theta}_0)$ are sequences of a white noise process in the time parameter. However, it does not imply that the prediction errors also have to be white noise in the spatial parameter. If $\overline{\epsilon}(t, \overline{\theta}_0)$ is white noise in spatial parameter $\overline{\overline{\Lambda}}_0$ becomes a diagonal matrix.

In the following it will be explained how $\overline{\overline{\Lambda}}_0$ is modelled in this example.

First, it is assumed that the measuring errors $\overline{e}_0(t)$ are given by

$$\overline{e}_0(t) = \overline{e}_1(t) + \overline{e}_2(t) \tag{6.161}$$

where $\overline{e}_1(t)$ models the error due to the random fluctuations of the system and $\overline{e}_2(t)$ models the random measuring errors. The random measuring error $\overline{e}_2(t)$ models, as mentioned above, the noise caused by e.g. an unmeasured excitation or that the sensors are subjected to noise. The noise can also be caused by the model giving an uncorrect description of the system under consideration. This random measuring error is normally modelled as a white noise sequence both in time and space parameters. As mentioned before, this is generally not valid. If, for example the noise is caused by an unmeasured excitation, the observed response to the noise at one point in a structure will be correlated with other points in the structure.

To investigate $\overline{e}_1(t)$ closer it is assumed that the measurements are not encumbered with random measuring noise, i.e.

$$\overline{e}_0(t) = \overline{e}_1(t) \quad \wedge \quad \overline{e}_2(t) = 0 \tag{6.162}$$

If a correct model is known for the system and the excitation is deterministic it is possible to calculate the output exactly. Then the measuring error $\overline{e}_1(t)$ is zero, i.e. the result is not encumbered with noise.

Since a random excitation is used in this example there will be a measuring error implying $\overline{e}_1(t) \neq 0$. The error $\overline{e}_1(t)$ is obtained since the measured response is a realization of a stochastic process. However, the output calculated by a correct model for the system will also be a realization of a stochastic process. These two realizations of the output, the measured and the calculated output, will be different and therefore, an error is obtained.

After it is explained that $\overline{e}_1(t) \neq 0$ in this example it will be explained how the covariance matrix of the noise process is modelled.

From above it is seen that a realization $\overline{e}_1(t)$ of the stochastic process $\{\overline{\mathcal{E}}_1(t)\}$ has to be a realization of an independent zero mean vector.

The structure considered in this example is subjected to white noise which is filtered by the model of the structure implying that the response may be correlated in time thus, the noise $\overline{e}_1(t)$ may also be correlated. However, it is assumed in this example that the noise $\overline{e}_1(t)$ can be modelled as a sequence of a zero mean vector.

Next, it is explained why $\overline{e}_1(t)$ is assumed to be correlated in the space parameter z and how the correlation is taken into account.

In (6.93) it is seen that the response of a linear system can be estimated by a modal approach. If only the response of the first mode is considered it is seen that the difference between the responses at two different points is due to the mode shapes. The two responses are the same sequence except a constant since they are produced by the same modal response $q_1(t)$. Thus, there will be a correlation between the measurements at different points since the responses are coupled by the mode shapes. This implies that the correlation of the noise $\overline{e}_1(t)$ in the space parameter z corresponds to the correlation of the response $y(z,t)$.

In this example this result implies that the covariance of the noise process $\{\overline{\mathcal{E}}_1(t)\}$ is modelled as a white noise process in the time parameter and assumed to be correlated in the space parameter corresponding to the correlation of the response process between different points. This means that $\overline{\overline{\Lambda}}_0$ can be obtained from the correlation matrix $\overline{\overline{R}}$ by a scaling. However, if it is only the optimal locations of the sensors which are of interest and not the magnitude of the elements in the parameter covariance matrix $\overline{\overline{P}}_{\hat{\overline{\theta}}_N}$ the correlation matrix $\overline{\overline{R}}$ can be used in (6.160) instead of the correct covariance matrix of $\overline{\overline{\Lambda}}_0$ of the noise process $\overline{\mathcal{E}}_1(t)$. If the correct covariance matrix $\overline{\overline{\Lambda}}_0$ is of interest the elements in the matrix can be estimated by calibrating $\overline{\overline{P}}_{\hat{\overline{\theta}}_N}$ using the Fisher information matrix. E.g. if one measuring point is considered then the following relation has to hold

$$\det \overline{\overline{J}}^{-1} = \det \overline{\overline{P}}_{\hat{\overline{\theta}}_N} = E[\overline{\overline{\Psi}}(t,\overline{\theta}_0)\overline{\overline{\Psi}}^T(t,\overline{\theta}_0)]/\lambda_1 \tag{6.163}$$

where λ_1 is the variance of the noise process $\{\overline{\mathcal{E}}_1(t)\}$. It may be remembered that this process models the error due to random fluctuations. (6.162) implies that λ_1 can be determined corresponding to a given measuring point.

If two measuring points are considered it is the elements in $\overline{\overline{\Lambda}}_0$ which have to be calibrated. Above it is explained that the correlation of the noise $\overline{e}_1(t)$ in the space parameter z corresponds to the correlation of the response $y(z,t)$. This means that the correlation coefficient ρ_{12} between two points is given by

$$\rho_{12} = \frac{R_{YY}(z_1,z_2;0)}{\sqrt{R_{YY}(z_1,z_1;0)R_{YY}(z_2,z_2;0)}} = \frac{\lambda_{12}}{\sqrt{\lambda_1\lambda_2}} \tag{6.164}$$

where λ_{12} is the covariance of the noise process. Since ρ_{12} is known it is seen from (6.164) that λ_{12} can be determined when λ_1 and λ_2, respectively, have been determined from (6.163). If more than two points are considered the elements in $\overline{\overline{\Lambda}}_0$ can be determined as described above for two points. This calibration will be used in an example in chapter 8.

The applicability of using (6.160) for the determination of the optimal locations of sensors is investigated in the following.

In figure 6.19 the determinant of the inverse of the parameter covariance matrix $\det \overline{\overline{P}}^{-1}_{\hat{\overline{\theta}}_N}$ is shown as a function of the locations of two sensors and it is assumed that

the measurements are taken only once at each point. The parameter covariance matrix $\overline{\overline{P}}^{-1}_{\hat{\theta}_N}$ given by (6.160) is estimated by describing the response of the beam by a modal approach. Here, it will not be explained in detail how the covariance matrix is estimated.

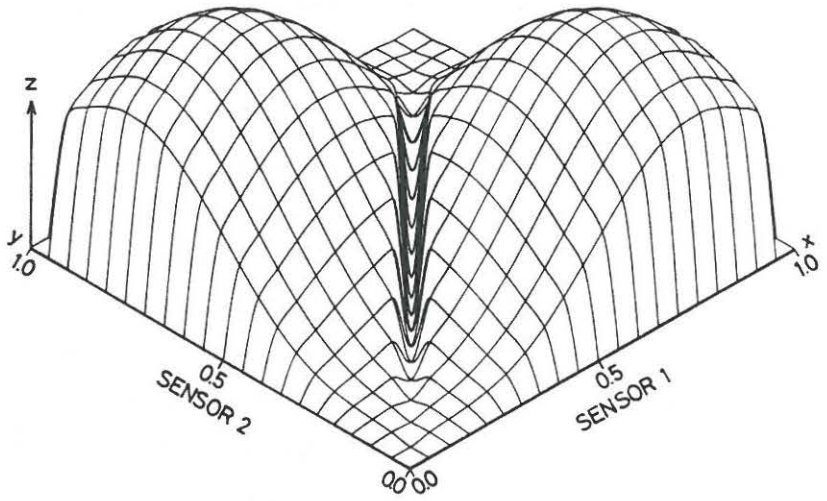

Figure 6.19: The determinant of the inverse of parameter covariance matrix $\overline{\overline{P}}^{-1}_{\hat{\theta}_N}$ shown as a function of the locations of two sensors.

It is seen that figure 6.19 corresponds to figure 6.7 obtained by using the Fisher information matrix. Further, it is seen, see table 6.3, that the optimal locations of the two sensors almost correspond to the optimal locations found by using the Fisher information matrix. One could have expected that the results would have coincided since the locations are found from the mode shape information. If one point is considered the optimal location is $0.865L$.

N_s	z_1	z_2	z_3	z_4	z_5
1	0.866L				
2	0.265L	0.735L			
3	0.264L	0.633L	0.799L		
4	0.187L	0.354L	0.646L	0.813L	
5	0.154L	0.289L	0.416L	0.653L	0.818

Table 6.3: Locations for different number of sensors. L is the beam length.

In section 6.4.2 it is seen that it is cumbersome to calculate the Fisher information matrix for more than a two-dimensional problem. By using (6.160) it is easy to estimate the optimal locations of more than two sensors. Further, it is also easy to estimate the parameter covariance matrix if $N > 1$, i.e if it is assumed that the measurements are taken more than once at each point.

In figure 6.20 the determinant of the inverse of the parameter covariance matrix $\det \overline{\overline{P}}^{-1}_{\hat{\theta}_N}$ is shown as a function of the number of sensors N_s and the number of data N.

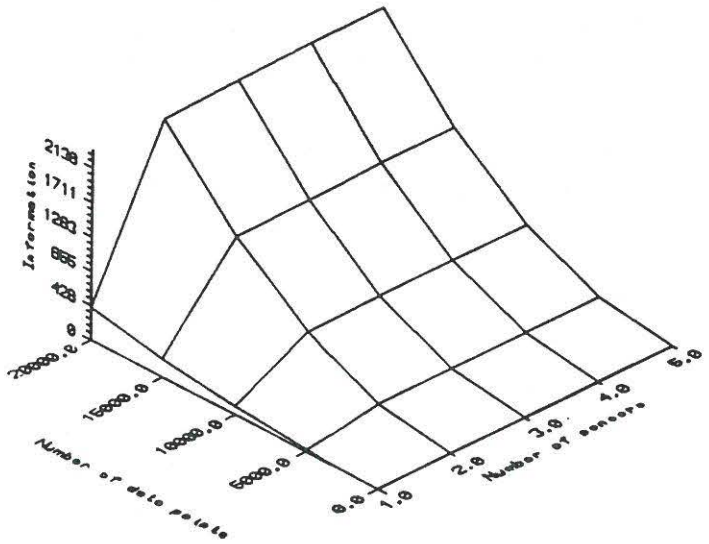

Figure 6.20: The determinant of the inverse of the parameter covariance matrix $\det \overline{\overline{P}}^{-1}_{\hat{\theta}_N}$ shown as a function of the number of sensors N_s and the number of data N.

It is interesting to note from figure 6.20 that the contribution of additional sensors beyond 2 does not substantially improve the experiment design. Further, it is seen, as expected, that for an increasing amount of data the uncertainty is decreasing. The results imply that the following question can be asked, again

- What is the optimal number of sensors N_s and data N in each time series ?

It is a question which seems unsolved in the papers dealing with optimal experiment design. In chapters 7 and 8 a method is proposed to answer the question.

It may be noticed that the results shown in this section are estimated by assuming $\overline{e}_2(t) = 0$, i.e. the random measuring noise is assumed to be zero. However, real measurements are encumbered with random measuring noise implying $\overline{e}_2(t) \neq 0$. In section 6.4.2 the influence of the random measuring noise is considered and will not be repeated here. One may expect the same conclusions as in section 6.4.2.

6.5 Summary

In this chapter different methods proposed for determining optimal locations of sensors for the parametric identification of structural systems have been considered and evaluated. The summary of the chapter and the conclusions of the evaluation of the different methods can be stated as follows:

- Design of an optimal experiment on purely heuristic grounds may be difficult since simple examples considered in this chapter have yielded results showing that the optimal locations of sensors appear to depend under circumstances in a more or less complex manner on the actual parameter values of the system and the excitation.
- It is evident from the examples that the experimental conditions have an effect on the achievable accuracy. Thus, there is a motivation in practice to choose the appropriate sensor locations to optimize the information return from the experiment.
- It is seen that the optimization problem can be flat near the minima. This causes difficulties in the precise choice of the optimal locations but it also means that some imperfections in the design or the practical positioning of the sensors results in a relatively small increase of error. This is a result which seems also to apply if other design variables are considered.
- The optimal locations of sensors seem to become less sensitive to e.g. the noise-to-signal ratio for increasing number of sensors. This result also seems to apply if the sensitivity of the optimal location of sensors is investigated with regard to other variables which enter into the problem.
- The concept of entropy is found to fail when it is used for determining the optimal locations of sensors.
- It is seen that there are very few papers devoted to the case of design of experiments in which the subject of measurements is modelled by the random field with non-trivial covariance function. However, when a continuous structural system is subjected to random load the measurements can hardly be modelled as statistically independent random variables, since the noise, at different measuring points, due to the random fluctuations is coupled through the mode shapes.
- The question: "What is the optimal number of sensors ?", seems unsolved in the papers dealing with optimal experiment design. To answer the question it is necessary to take into account the cost of using an additional sensor. Further, it is also necessary to consider the increase in the information value by using an additional sensor.

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Chapter 7

Updating of Structural Reliability by Performing System Identification Experiments

In the foregoing chapters various methods available for design of parameter identification experiments have been discussed. By using these methods it is possible to design experiments where the quality of the design is expressed by a scalar measure of an expected estimated parameter covariance matrix. Such a quality measure can be used to compare different designs, but it cannot tell, how much one design is better than another design. This means, as it is seen above, that e.g. design of an experiment concerning optimal location of sensors is possible, but it is not possible to estimate the optimal number of sensors. More general, it can be said that the design approaches discussed in the preceding chapters do not make it possible to investigate the increase in the value of the information expected to be obtained by changing an experiment design in direction of a more optimal design. This implies that the acquisition of additional information by performing a full-scale measuring of a structure can result in unnecessary use of resources. This is because time, energy and financial resources are not reflected in the experiment design. However, if these quantities should be reflected in the design, the expected utility of performing the experiment should be known, before the experiment is performed. If this utility is expressed in a monetary value it will be possible to make a tradeoff study between the increased monetary value and the cost of performing the experiment.

In chapter 8 it will be explained how a system identification experiment can be designed based on such a tradeoff study. The utility in monetary values expected to be obtained by performing an experiment will be expressed by the expected updated structural reliability. When experiments are performed additional information about the uncertain parameters is obtained. This implies that the structural reliability is changed, caused of a reduction by the uncertainty.

In this chapter the structural reliability theory will be presented in section 7.1. In section 7.2, an example is given to show how the theory can be used, and it will be shown that modern reliability methods are also an excellent tool to determine important sources of uncertainty. After that, in section 7.3, it is explained how the updated structural reliability can be estimated when additional information obtained from system identification experiments becomes available. At last in section 7.4 and 7.5, respectively, examples are given to show how the updated structural reliability can be estimated as a function of the system identification experiment design variables. It is shown that an updated structural reliability can be used as an experiment design criterion.

7.1 Structural Reliability Theory.

In traditional design of civil engineering structures, such as offshore structures, bridges etc. which often exhibit a significant dynamic amplification of response the expected values of e.g. mass, damping, stiffness properties are used. When the dynamic amplification becomes important also the uncertainty in the mass, damping and stiffness properties becomes an important task to describe. A study of the effect of structural parameter uncertainty is conveniently done by use of modern reliability methods. Modern reliability methods have been extensively applied in the last decade, where considerable progress has been made in the area of structural reliability theory. Especially, the development of the so-called first-order reliability methods (FORM) and the second-order reliability methods (SORM) have been very important, see e.g. Madsen et al. [1], Thoft-Christensen et al. [2], Ditlevsen et al. [3]. These methods are especially developed to estimate the reliability of structural elements and systems. These reliability methods are also an excellent tool to determine important sources of uncertainty.

7.1.1 Element Reliability

A reliability analysis is based on a reliability model of the structural system. The elements in the reliability model are failure elements, modelling potential failure modes of the structural system, e.g. fatigue failure, yielding failure, buckling failure etc. Each failure element is described by a failure function $g(\overline{x}, \overline{p}) = 0$ in terms of a realization $\overline{x}$ of a random vector $\overline{X} = (X_1, X_2, .., X_{n_X})$, and deterministic parameters $\overline{p}$, i.e. deterministic design parameters and parameters describing the stochastic variables, (expected value and standard deviation). $\overline{X}$ is assumed to contain n_X stochastic variables, e.g. variables describing the loads, strength, geometry, model uncertainty etc. Realizations $\overline{x}$ of $\overline{X}$ where $g(\overline{x}, \overline{p}) \leq 0$ correspond to failure states in the n-dimensional basic variable space, while $g(\overline{x}, \overline{p}) > 0$ correspond to safe states.

The reliability R of the failure element can now be written

$$R = 1 - P_f = 1 - \int_{g(\overline{x},\overline{p})\leq=0} f_{\overline{X}}(\overline{x})d\overline{x} \approx 1 - \Phi(-\beta) \tag{7.1}$$

where P_f is the probability of failure. $f_{\overline{X}}(\overline{x})$ is the joint probability density function of $\overline{X}$ and $\Phi(\cdot)$ is the one-dimensional standard normal distribution function.

In first-order reliability methods (FORM) the approximation in (7.1) is obtained by using a transformation $\overline{T}$, see e.g. Madsen et al. [1] of the generally correlated and non-normally distributed variables $\overline{X}$ into standardized, normally distributed variables $\overline{U} = (U_1, U_2, .., U_{n_X})$ is defined. Let $\overline{U} = \overline{T}^{-1}(\overline{X}, \overline{p})$. In the $\overline{u}$-space the reliability index β is defined as

$$\beta = \min_{g(\overline{T}(\overline{u}),\overline{p})=0} (\overline{u}^T\overline{u})^{\frac{1}{2}} \tag{7.2}$$

The solution point $\overline{u}^\star$ of the optimization problem in (7.2) is the point on the failure surface $g(\overline{x}, \overline{p})$ closest to the origin in the $\overline{u}$-space and is called the design point, or β-point. The reliability index β is thus determined by solving an optimization problem with one constraint. The optimization problem is generally non-linear and can in principle be solved using any general non-linear optimization algorithm, but the iteration algorithm developed by Rackwitz and Fiessler, see Madsen et al. [1], is traditionally used in FORM since it has shown to be fast and effective in FORM analysis.

It is seen that the reliability index β is introduced as a measure of the reliability which can be estimated based only on second moment information of the uncertainties entering the reliability problem.

In FORM the safety margin M defined by

$$M = g(\overline{T}(\overline{U}), \overline{p}) \tag{7.3}$$

is linearized in the design point

$$M \approx -\overline{\alpha}^T\overline{U} + \beta \tag{7.4}$$

where the elements in the $\overline{\alpha}$-vector

$$\alpha_i = \frac{u_i^*}{\beta} = \frac{-1}{|\nabla_{\overline{u}} g|}\frac{\partial g}{\partial u_i} \tag{7.5}$$

where $\nabla_{\overline{u}} g$ is the gradient of g with respect $\overline{u}$ in the design point $\overline{u}^\star$. It is assumed that the first order derivative of the failure function exist. The probability of failure P_f can now according to (7.1) be approximately determined from

$$P_f \approx \Phi(-\beta) \tag{7.6}$$

It should be noticed that a better reliability estimate can be obtained by an improved approximation of the failure surface. A quadratic approximation of the failure surface at the design point is called a second-order reliability method (SORM). Computation of SORM estimates can be costly when $\overline{X}$ is large and the failure function involves complicated numerical algorithms, e.g. finite element analysis, numerical integration etc. because the second order derivatives at the design point are required in SORM.

In the foregoing it is explained how the reliability can be estimated when a structure is modelled by a single failure element. In the following sections it is explained how the reliability of structural systems can be modelled by series and parallel systems.

7.1.2 Series System Reliability

If the whole structural system is modelled, as a series system, by m failure elements, and failure of the system is defined as failure of one failure element, then the probability of failure can be determined by

$$P_f = P(\bigcup_{i=1}^{m}\{g_i(\overline{x},\overline{p}) \leq 0\}) \tag{7.7}$$

If the system has failure elements described by linearized safety margins (7.4) then (7.7) can be estimated as

$$P_f \approx P(\bigcup_{i=1}^{m}\{\overline{\alpha}_i^T\overline{U} + \beta_i \leq 0\}) = 1 - \Phi_m(\overline{\beta},\overline{\overline{\rho}}) \tag{7.8}$$

where $\overline{\beta} = (\beta_1, \beta_2, .., \beta_m)^T$ are the reliability indices of the failure elements. $\overline{\overline{\rho}}$ is the corresponding correlation coefficient matrix for the linearized safety margins given by

$$\rho_{ij} = \overline{\alpha}_i^T\overline{\alpha}_j \qquad i,j = 1,2,...,m \tag{7.9}$$

$\Phi_m(\cdot)$ is the m-dimensional standard normal distribution function.

A generalized systems reliability index β^s of this series system can be estimated from

$$\beta^s = -\Phi^{-1}\left(1 - \Phi_m(\overline{\beta};\overline{\overline{\rho}})\right) \tag{7.10}$$

Since numerical calculation of the multi standard normal function $\Phi_m(\cdot)$ is very time consuming or more or less impossible for large values of m several different methods have been developed to make approximately calculations such as the Ditlevsen bounds, the simple bounds, average correlation coefficient approximation and the Hohenbichler approximation, see e.g Thoft-Christensen et al. [2].

7.1.3 Parallel Systems Reliability

If the whole structural system is modelled, as a parallel system, by m failure elements, then the system is considered to be in failure state when all the failure elements fail.

The probability of failure is determined as

$$P_f = P(\bigcap_{i=1}^{m}\{g_i(\overline{x},\overline{p}) \leq 0\}) \tag{7.11}$$

If the system has failure elements described by linearized safety margins (7.4) then (7.11) can be estimated as

$$P_f \approx P(\bigcap_{i=1}^{p}\{\overline{\alpha}_i^T\overline{U} + \beta_i \leq 0\}) = \Phi_p(-\overline{\beta},\overline{\overline{\rho}}) \tag{7.12}$$

where $\overline{\beta}$ is the vector with reliability indices and $\overline{\overline{\rho}}$ is the correlation coefficient matrix of the failure elements in the parallel system, respectively, corresponding to the $p \leq m$ number of active constraints in the following optimization problem $\overline{u}^*$

$$\begin{array}{rl} min & |\overline{u}| \\ s.t & g_1(\overline{u}) \leq 0 \\ & g_2(\overline{u}) \leq 0 \\ & .... \\ & g_m(\overline{u}) \leq 0 \end{array} \tag{7.13}$$

The m number of failure functions $g_i(\overline{u})$ corresponds to the safety margins in (7.12). The optimization problem in (7.13) can be solved by using standard optimization techniques and gives the joint design point $\overline{u}^\star$. In Enevoldsen [4] an algorithm (JOINT3) is proposed which seems to be fast and stable compared to the standard optimization techniques.

A generalized systems reliability index β^p of this parallel system can be estimated from

$$\beta^p = -\Phi^{-1}\left(\Phi_p(-\overline{\beta};\overline{\overline{\rho}})\right) \tag{7.14}$$

A more thorough description of the estimation of the probability of failure of the parallel system by using FORM can be found in e.g. Madsen et al. [1], Ditlevsen et al. [3], Enevoldsen [4] and Enevoldsen et al. [5].

It may be noticed that it is also possible to have a series system of parallel systems. The reliability of such a system can be estimated by estimating an equivalent reliability-index and α-vector for an equivalent linear failure element for each parallel system. Then, based on the equivalent linear failure elements the system

reliability β^s can be estimated using (7.10). Significant parallel systems can be identified using e.g. a so-called β-unzipping method. see e.g. Thoft-Christensen et al. [2] or a branch and bound technique, see e.g. Guenard et al. [6].

7.1.4 Sensitivity Analysis

Besides the absolute values of the element reliability indices β_i and the systems reliability index β^s , it is often of interest to know the sensitivity of the element reliability indices and the systems index to variations of parameters $\overline{p}$. $\overline{p}$ is a parameter vector including statistical parameters (mean value and standard deviation) describing the random variables in $\overline{X}$.

The derivatives of β_i and β^s can be estimated in the following way, see e.g. Sørensen [7].

The derivative of the system reliability index is obtained by differentiating (7.10)

$$\frac{\partial \beta^s}{\partial p_j} = \sum_{i=1}^{m} \frac{\partial \beta^s}{\partial \beta_i} \frac{\partial \beta_i}{\partial p_j} + 2 \sum_{i<k}^{m} \frac{\partial \beta^s}{\partial \rho_{ik}} \frac{\partial \rho_{ik}}{\partial p_j} \tag{7.15}$$

where

$$\frac{\partial \beta^s}{\partial \beta_i} = \frac{\varphi(\beta_i)}{\varphi(\beta^s)} \Phi_{m-1}(\overline{\beta}_i^a; \overline{\overline{\rho}}_k^a) \tag{7.16}$$

$$\frac{\partial \beta^s}{\partial \rho_{ik}} = \frac{\varphi(\beta_i, \beta_k; \overline{\overline{\rho}}_{ik})}{\varphi(\beta^s)} \Phi_{m-2}(\overline{\beta}_{ik}^b; \overline{\overline{\rho}}_{ik}^b) \tag{7.17}$$

It is assumed that the m significant failure modes are numbered $1, 2, ..., m$. $\overline{\beta}_i^a$, $\overline{\overline{\rho}}_i^a$ $\overline{\beta}_{ik}^b$ and $\overline{\overline{\rho}}_{ik}^b$ are the conditional reliability indices and correlation coefficients, respectively, see Sørensen [7]. $\frac{\partial \rho_{ik}}{\partial p_j}$ can be determined as described in Bjerager et al. [8]

The derivative of the element reliability index $\frac{\partial \beta_i}{\partial p_j}$ follows from, see e.g. Madsen et al. [1]

$$\frac{\partial \beta_i}{\partial p_j} = \frac{1}{\beta} \sum_{l=1}^{nx} u_l^\star \frac{\partial \{T_l^{-1}(\overline{x}^\star, \overline{p})\}}{\partial p_j} \tag{7.18}$$

However, if the derivative of the element reliability index has to be determined with respect to a deterministic parameter p', not included in $\overline{p}$, then (7.19) has to be used

$$\frac{\partial \beta_i}{\partial p'} = \frac{1}{|\nabla_{\overline{u}} g_i|} \frac{\partial g}{\partial p'} \tag{7.19}$$

The gradients of the reliability index which are generally time-consuming to estimate numerically can be determined semi-analytically. It is seen from (7.18) and (7.19) that element reliability sensitivity can generally easily be solved because the reliability gradients can be easily determined by the simple expression.

$\frac{\partial \beta^*}{\partial p_j}$ can be approximated with a sufficient degree of accuracy by neglecting the correlation terms in (7.15) according to Sørensen [7] and Sørensen [9] . However, this implies that convergence problems can be expected if the gradients are used in relation to a mathematical programming algorithm. It may also be noticed that reliability based optimization where $\frac{\partial \beta^*}{\partial p_j}$ is determined by (7.15) can be expected to be rather costly. Further the system reliability index is generally estimated by an approximation. This implies that the estimates of $\frac{\partial \beta^*}{\partial p_j}$ based on these approximations do not have accuracy necessary for optimization. Therefore numerical derivatives may be used implying that the computation times become unacceptably large. Because of that alternative optimization procedures have been proposed where the time-consuming approximation of the system reliability index can be avoided. Such procedures are derived in e.g. Sørensen [10] and Enevoldsen et al. [4]. In the latter a system modelled as a series system of parallel systems is considered and the expression similar to (7.15) are established for sensitivity analysis of a parallel systems and series system of parallel systems.

In the following example it is shown how the reliability methods can be used to estimate the reliability of a civil engineering structure. Further, it is shown how the reliability methods can be used as an excellent tool to determine important sources of uncertainty.

The reliability calculations in this thesis are performed with the computer program PRADSS (Program for Reliability Analysis and Design of Structural Systems), see Sørensen [11].

7.2 Example 7.1: Reliability Analysis of a Civil Engineering Structure

In this example, a fatigue reliability analysis of a Mono-tower platform is presented. Element reliability as well as systems reliability are estimated using first-order reliability methods (FORM). The sensitivity of the systems reliability to various parameters is investigated.

In an analysis by Enevoldsen et al. [12] the surrounding soil and the tube cross-sections of the mono-tower are investigated for fatigue as well as yielding failure. According to the conclusions two major failure modes are significant. Firstly, yielding failure of the tube cross-sections due to an extremely high wave and secondly, fatigue failure in the circumferential butt welds along the mono-tower. In Kirkegaard et al. [13] and Kirkegaard et al. [14] the fatigue failure mode is investigated closer. The following is based on these two papers.

7.2.1 Description of Mono-Tower Platform

The Mono-tower structure, considered, has been described in Petersen et al. [15],

where data for the environmental conditions also have been stated. Originally, the structure had been designed as an attractive solution for a marginal oil and gas field (Rolf field) in the Danish Sector of the North Sea, but the plans for this field were changed to a traditional 4 - legs jacket structure.

The single pile platform, Mono-tower, investigated, is a remotely operated platform, with provision for four wells, designed for 33.7 m. of water in the Danish part of the North sea. The platform is a single steel cylinder driven into the seabed, supporting a topside facility deck. The structure has three different sections, with different diameters d, see figure 7.1. The wall thickness t of the Mono-tower platform is 80 mm; except a 7 m long, 100 mm thick, section from el. -4 to el.+3. The topside structure has an emergency deck at el. +15.6, a main deck at el. +19.0, a mezzanine deck at el. +21.7 and a helideck at el. +26.0. The total weight of the topside is 200 tons including the deck structure and all the equipment necessary for four wells. The total tower is weighting approximately 700 tons. The well conductors have been placed inside the pile, while an oil export riser, a ladder, a boat-landing and anodes have been placed outside the pile.

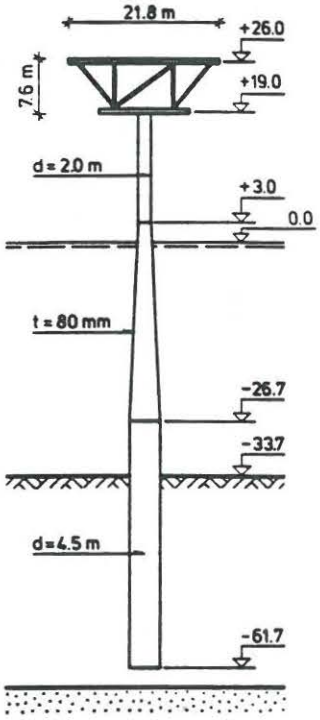

Figure 7.1: Elevation of Mono-Tower

7.2.2 Reliability Modelling of Mono-Tower

It is assumed that fatigue failure can occur only at the welded joints.

7.2.2.1 Fatigue Failure Element

The failure element corresponding to fatigue failure in the butt welds, is investigated with a failure $g(\overline{x}, \overline{p})$ with the fatigue strength expressed through SN relations. In determining the cumulative fatigue damage, Palmgren-Miner's rule is applied.

The failure function for the fatigue element is written

$$g(\overline{x}, \overline{p}) = D_{Fail} - (D_{Driving} + D_{wave}) \tag{7.20}$$

where D_{Fail} is the value of Palmgren-Miner's sum at failure. $D_{Driving}$ is the damage from the driving of the Mono-tower into the seabed and D_{Wave} is the damage from wave action.

The cumulative fatigue damage D_{wave} due to wave action is assumed to be given by the Palmgren-Miner's rule, where experimentally determined SN-curves are used to calculate the fatigue strength. It is assumed that the stress range at a time is double of the stress amplitude. Further, it is assumed that stress variation is a zero-mean narrow-band Gaussian process.

Under these assumptions, the total wave induced fatigue damage D_{wave} is calculated by summing up the mean fatigue damage per stress cycle within one sea state $\overline{D}_i$ over the service lifetime of the structure T_L, which is assumed to be 25 years, and weighting the mean fatigue damage for each sea state according to the long-term sea state probability density function for the significant wave height $f_{H_s}(h_s)$, which is assumed to be well represented by a Weibull density function. The coefficients in the Weibull distribution are estimated from a Wave-scatter diagram for the Danish part of the North sea. This leads to that the failure function (7.20) can be written

$$g(\overline{x},\overline{p}) = \ln(D_{Fail} - D_{Driving}) + \ln(K) - \ln(T_L) - k\ln(SCF2\sqrt{2}) - \ln(\Gamma(1+\frac{k}{2}))$$

$$-\ln(\int_0^\infty \int_{-\pi}^{\pi} \frac{(\sigma_s(h_s))^k}{T_0(h_s)} f_{H_s}(h_s) f_{\Phi_s}(\varphi_s) dh_s d\varphi_s) - \frac{k}{4}\ln(\frac{t}{22}) \qquad (7.21)$$

where $\Gamma(\cdot)$ is the gamma function. $\sigma_s(h_s)$ is the standard deviation of the stress response and $T_0(h_s)$ is the zero-upcrossing period of the stress cycles. $f_{\Phi_s}(\varphi_s)$ is the probability density function for the predominant wave direction. k and K are the parameters in the SN-curves to be determined from experimental data. Here, two different SN-curves are chosen by using criteria stated in Lotsberg et al. [16]. A so-called C-curve is used in the cone/cylinder transitions and below level -25.7. Otherwise, there is used a F2-curve. The SN-curves, used, have been intended for joints exposed to sea water and cathodic protected. The stress concentration factor SCF is assumed to be 1; expect at the cone/cylinder transitions, where SCF is calculated by a formula stated in API RP 2A [17]. Since the fatigue strength of welded joints decreases with increasing plate thickness t, see Berge [18], equation (7.21) has been corrected (the last term in (7.21)) for thicknesses different from than 22 mm, which the basic SN-curves have been related to.

7.2.2.2 Calculation of Structural Response

In order to estimate the statistical measures of stress variations, $\sigma_s^2(h_s)$, $T_0(h_s)$, the modal spectral analysis method is applied. It is assumed that the long-term sea state can be accurately modelled as a piecewise zero-mean stationary Gaussian process. Here, Pierson-Moskowitz sea spectrum is used. The transfer function from water elevation to wave forces on the Mono-tower is calculated by using linear Airy wave theory and Morison's equation, where the non-linear drag term is linearized by the "minimun square error method".

Hydrodynamic coefficients for the combined tube and riser have been estimated in Jacobsen et al. [19]. To take diffraction into account, the basic value for the inertia coefficient C_M is changed as function of the wave length. The structure is modelled as a one-dimensional, linear system with the two lowest natural frequencies, $f_1 = 0.49$ Hz., $f_2 = 2.19$ Hz.

7.2.2.3 Stochastic variables

In table 7.1, the statistical characteristics of the basic variables are fully enumerated. Further, there is shown the deterministic design parameters, which are investigated in a sensitivity analysis. In this paper, statistical characteristics of the basic variables for both the fatigue failure elements are mainly from published information. In Enevoldsen et al. [12], the stipulation of the statistical characteristics for the fatigue element has been discussed in details. The SI units system is used.

Variable	Designation	Distrib.	Exp. value	Var. Coeff
C_D	Drag coefficient	N	1.0*	0.2
C_M	Inertia coefficient	N	1.0*	0.2
TM	Mass of topside	N	200000	0.1
t	Wall thickness	N	1.0*	0.05
SCF	Stress concent. factor	N	1.0*	0.1
B	Parameters in long-term	N	2.35	0.1
C	distribution of H_s	N	1.89	0.1
$Equi$	Equivalent stiffness	N	1.0*	0.1
m_1	Thickness correction	LN	1.0*	0.1
λ	Coeff. for added mass	N	0.9	0.1
$D_{Driving}$	Damage from "driving"	LN	1.0*	0.15
ζ	Damping ratio	LN	0.015	0.5
K	Constant in SN-curve	LN	1.0*	0.65
D_{fail}	Damage at failure	LN	1.0	0.3
Z_1	Model uncertainty	N	1.0	0.2
d	Tube diameter	D	1.0*	
d_1	Marine growth	D	1.0*	
G	Acceleration of gravity	D	9.82	
ρ_w	Density of sea water	D	1025	
h	Water depth	D	33.7	

Table 7.1: Statistical characteristics (EX1 : Extreme type 1, N : Normal, LN : Lognormal, D : Deterministic)

Expected values represented by 1.0* indicate that the expected value varies along the structure. In the reliability calculations, the expected value 1.0* is multiplied with the real expected value of the stochastic variable at the given level. The expected value of TM includes permanent loads and not live loads. $m_1 = \frac{k}{4}$ model

the uncertainty with the plate thickness reduction factor. To take into account the uncertainty of the stiffness of the soil and structure, respectively, the equivalent stiffness *Equi* is modelled stochastic. A direct stochastic modelling of the stiffnesses is not possible, as the eigenvalue analysis has been excluded from the reliability calculations. Uncertainties in the calculation of added mass, due to surrounding water, are modelled by λ. Uncertainties of the different contributions to the damping of the structure are taken into account by modelling the modal damping ratio as a stochastic variable. Normally, it is assumed that the damping of a Mono-tower consists of structural damping, viscous hydrodynamic damping, radiation damping and soil damping. It is seen in table 7.1 that only K in the SN relation is modelled as a stochastic variable. It is proposed by Wirsching [20], where statistical characteristics of K are stated, too. D_{fail} is a model uncertain variable, which models the uncertainty connected by Palmgren-Miner's rule. The other model uncertainty variable Z_1 models the uncertainties connected by the models, which are used to calculate the variance and the zero-upcrossing period of the stress process. The statistical characteristics of this stochastic variable have been chosen according to Wirsching [20]. C_D and C_M are assumed to be mutually correlated with the correlation coefficient ρ = -0.9. All the other stochastic variables are assumed to be independent.

7.2.3 Results

The Mono-tower platform is modelled as a series system with eighteen fatigue failure elements between level -33.7 and +15, see figure 7.2.

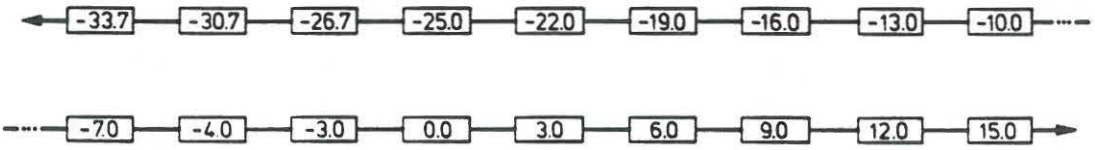

Figure 7.2: Location of fatigue elements.

Each element is assumed to model the damage at that point in the butt weld, where the greatest fatigue damage will occur. Between the failure elements, the stochastic variables modelling K are assumed to be correlated with the correlation coefficient ρ = 0.5. The same assumption is also made for D_{fail}. All the others stochastic variables are separately assumed fully correlated between the failure elements. The variation of the element reliability index β_i along the structure is shown in figure 7.3.

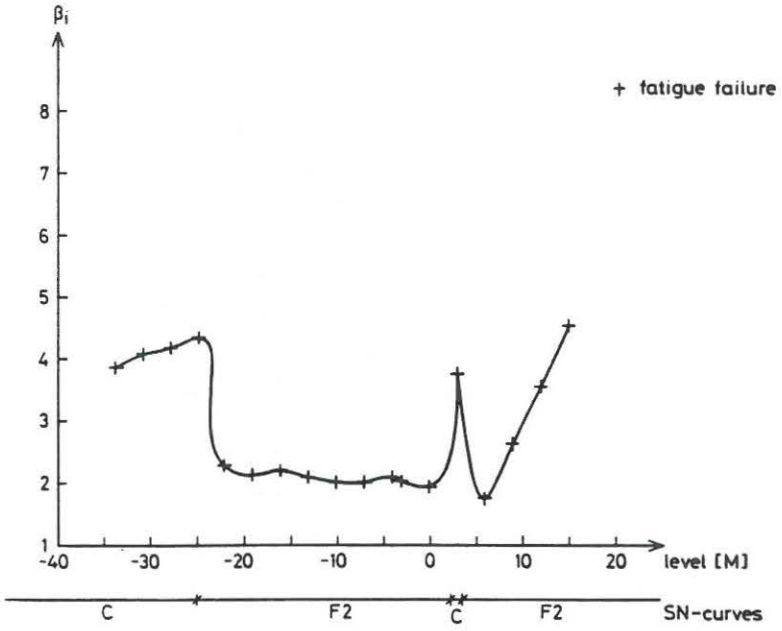

Figure 7.3: The variation of the element reliability index β_i along the Mono-tower platform. (Notice, the influence of the stipulation of SN-curves on β_i.)

Using the Hohenbichler approximation, the systems reliability index becomes $\beta^s = 1.432$

It is seen from figure 7.3 that the element reliability index is very sensitive to different SN-curves, as the reliability index is significantly changed, when the SN-curve is changed. This is also seen from the results of the sensitivity analysis, see figure 7.4.

Figure 7.4 shows the sensitivity of the systems reliability index β^s to variations of the expected values of the stochastic variables $\frac{\partial \beta^s}{\partial \mu_j}$ and standard deviations $\frac{\partial \beta^s}{\partial \sigma_j}$. The sensitivities can be relatively compared, as each derivative is multiplied by a hundredth parameter.

Figure 7.4 shows that many stochastic variables contribute to the overall uncertainty. Especially, K, C_M, SCF, ζ, D_{fail} and Z_1 contribute to the uncertainty. The systems reliability is also seen to be very sensitive to variations of the deterministic design parameters, except the marine growth.

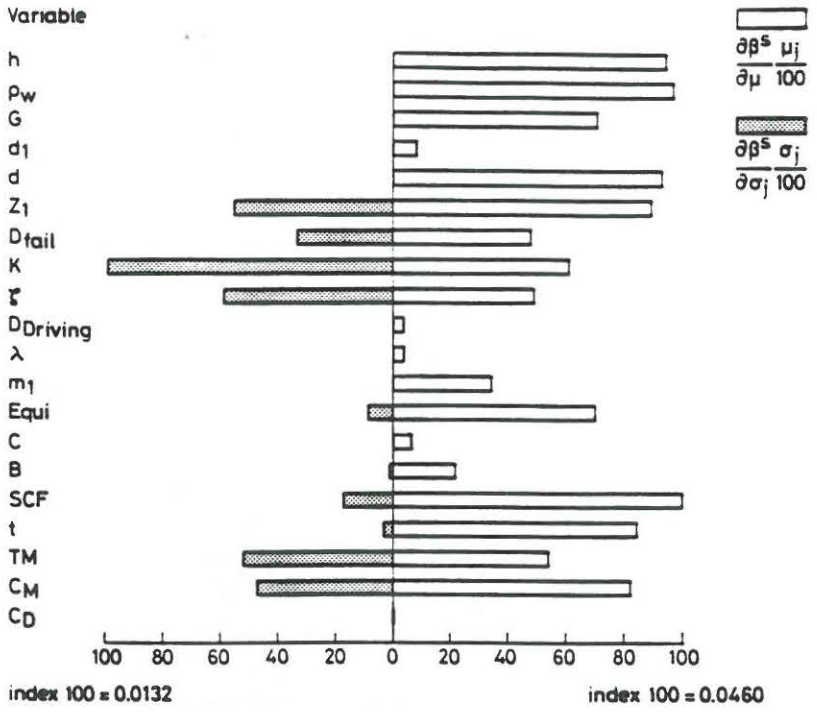

Figure 7.4: Sensitivity of the systems reliability to variations of the parameters of the stochastic variables.

The sensitivity of the systems reliability to variations of the modal damping ratio and natural period is shown in figure 7.5.

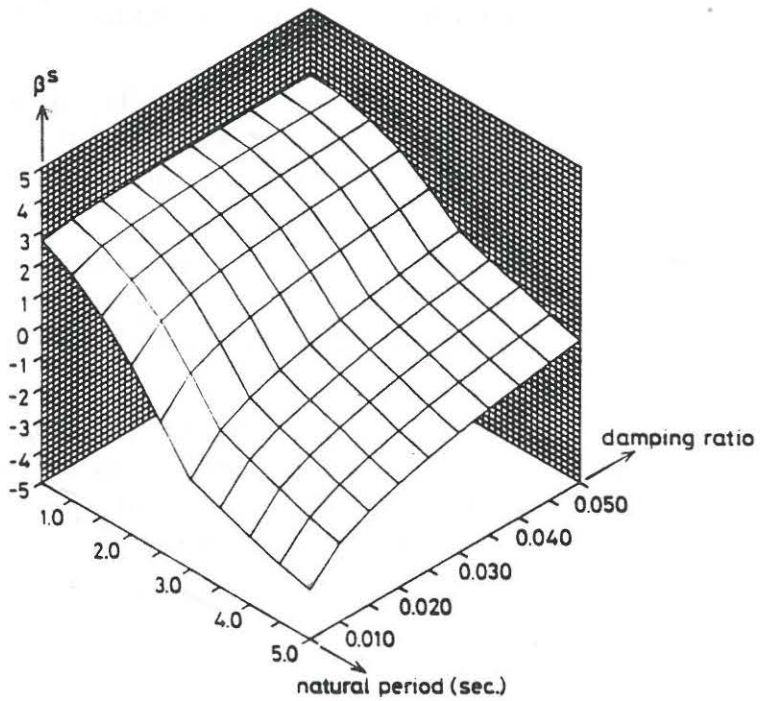

Figure 7.5: Sensitivity of the systems reliability index to variations of natural period and modal damping ratio.

Figure 7.5 proclaims that the systems reliability index for a Mono-tower platform with natural period greater than 1.5 sec. is very sensitive to variations of the damping ratio. Especially, for damping ratios less than 0.03-0.04.

7.2.4 Conclusions

Based on the reliability analysis of the Mono-tower platform the following conclusions can be stated:

1) Modern reliability methods can be used in an uncertainty analysis to calculate a nominal element reliability level as well as a systems reliability level. It has been shown, that the reliability methods can be used to estimate the sensitivity of the reliability in order to identify the most important uncertainties, by that pointing at problems for closer investigations.
2) A sensitivity analysis with respect to the systems reliability index showed that the largest contributions to the overall uncertainty are due to the damping ratio, the inertia coefficient, the stress concentration factor and parameters describing the fatigue strength.
3) For a Mono-tower platform, the systems reliability index is very sensitive to variations of the natural period and the damping ratio.

7.3 Updating of Structural Reliability

In the above example it is shown that modern reliability methods are well suited to analyse a problem where many parameters contribute to the total uncertainty. Such an analysis indicates where further studies can be useful to reduce the uncertainty about each parameter. In the following it will be shown that the modern reliability methods also provide a rational tool for updating a reliability analysis when additional information becomes available. First, it is explained how additional information can be incorporated into the reliability analysis. After that, it is shown how the reliability can be updated based on vibration measurements.

Updating of a structural reliability analysis means to couple additional information from fabrication and service of a structure to the design information, prior information. Service information is gotten from proof loading, inspection, vibration measurements, repair etc.. The updating of the reliability analysis can be used when decisions have to be made concerning e.g. extension of life time, inspection planning etc. Examples of applications can be found in e.g. Diamantidis [21] where reliability assessment of existing offshore structures is considered and Madsen et al. [22] consider inspection planning. The great interest for development of methods for estimatimation of the reliability of existing structures is caused by the fact that much of evaluation of the safety of existing structures is based on the engineer's judgement.

The theory of updating of a reliability analysis is given in e.g. Madsen [1] and Ditlevsen et al. [3]. A part of the additional information can directly be related to the basic variables $\overline{X}$ as

- sample information,(updating of the description of random variables).

Parts of additional information can also be related to a functional relation between basic variables. If the mass and stiffness of a structure are modelled as random variables, in a reliability analysis, then the eigenfrequency is related to the stiffnes and mass by a functional relation. Such additional information is called:

- relation information.

The problem by updating of a reliability analysis is to combine the new information with the prior information to obtain a posterior information. In section 7.3.1 the most traditional method available to update prior information based on new information is described.

7.3.1 Updating of the Description of Random Variables

Normally, it is assumed that the random vector $\overline{X}$ models the following four sources of uncertainty: Inherent variability, estimation error, model imperfection and human error.

Inherent variability, often called randomness, may exist in the characteristics of the structure itself or in the environment to which the structure is exposed.

Estimation error arises from the incompleteness of statistical data and our inability to accurately estimate the parameters of the probability models that describe the inherent variabilities.

Model imperfection arises from our use of idealised mathematical models to describe complex phenomena, and finally, the human error uncertainty arises from errors made by engineers or operators in the design, construction or operation phases of the structure.

Inherent variability is essentially a state of nature and the resulting uncertainty may not be controlled or reduced, i.e. the uncertainty associated with inherent variability is something which cannot be reduced. The uncertainty associated with estimation error, model imperfection and human error may be reduced through the acquisition of additional data, the use of more accurate models and implementing rigorous quality control measures in the design, construction and operation phases of a structure.

The available statistical information, objective and subjective, on relevant variables and the set of mechanical and probabilistic models and their associated error

estimates constitute the state of knowledge in a reliability problem. The state of knowledge is said to be perfect when complete statistical information and perfect models are available; otherwise, the state of knowledge is said to be imperfect. Real engineering problems invariably deal with imperfect states of knowledge.

How to deal with the four sources of uncertainty in a reliability analysis is explained in e.g. Ditlevsen [23], Ditlevsen [24], Ditlevsen et al. [3] and Veneziano [25]. In the following it is repeated how additional information can update the information of the random vector $\overline{X}$. It is the problem of including statistical uncertainty in a structural reliability analysis.

First, it is assumed that the additional information is a sample $\overline{x}_i$ of k observations of $\overline{X}_i$. The sample $\overline{x}_i$ is considered as an observation of the k-set $\overline{X}_i$ of mutually independent uncertainty quantities. A density function is assigned to the random vector $\overline{X}$

$$f_{\overline{X}|\overline{\Gamma}}(\overline{x}|\overline{\gamma}) \tag{7.22}$$

where $\overline{\gamma}$ is a realization of a random vector $\overline{\Gamma}$ including parameters in the density function, e.g. mean values and standard deviations. This assumption implies that the corresponding conditional $(n+k)$-dimensional density is

$$f_{\overline{X}_1,...,\overline{X}_k|\overline{\Gamma}}(\overline{x}_1,...,\overline{x}_k|\overline{\gamma}) = \prod_{i=1}^{k} f_{\overline{X}|\overline{\Gamma}}(\overline{x}_i|\overline{\gamma}) \tag{7.23}$$

Based on the prior information about $\overline{\Gamma}$ a prior density $f_{\overline{\Gamma}}(\overline{\gamma})$ is assigned to $\overline{\Gamma}$. The total density of $\overline{X}_i, i = 1,...,k$ and Γ becomes

$$f_{\overline{X}_1,...,\overline{X}_k}(\overline{x}_1,...,\overline{x}_k;\overline{\gamma}) = \prod_{i=1}^{k} f_{\overline{X}|\overline{\Gamma}}(\overline{x}_i|\overline{\gamma}) f_{\overline{\Gamma}}(\overline{\gamma}) \tag{7.24}$$

and a conditional density for $\overline{\Gamma}$ is obtained by use of Bayes' rule

$$f_{\overline{\Gamma}}(\overline{\gamma}|\overline{x}_1,...,\overline{x}_k) \propto L(\overline{\gamma};\overline{x}_1,...,\overline{x}_k) f_{\overline{\Gamma}}(\overline{\gamma}) \tag{7.25}$$

The left hand side is the posterior density of the parameters $\overline{\Gamma}$ which is proportional to the product of the likelihood function $L(\overline{\gamma};\overline{x}_1,...,\overline{x}_k)$ and the prior density $f_{\overline{\Gamma}}(\overline{\gamma})$. This product simply represents the total information available about the parameters after sampling.

It is seen from the definition of the likelihood function that

$$L(\overline{\gamma};\overline{x}_1,...,\overline{x}_s,\overline{x}_{s+1},...,\overline{x}_{s+t}) = L(\overline{\gamma};\overline{x}_{s+1},...,\overline{x}_{s+t}) L(\overline{\gamma};\overline{x}_1,...,\overline{x}_s) \tag{7.26}$$

This means that the posterior probability density $\overline{\Gamma}$ can be written as

$$f_{\overline{\Gamma}}(\overline{\gamma}|\overline{x}_1,...,\overline{x}_s,\overline{x}_{s+1},...,\overline{x}_{s+t}) \propto L(\overline{\gamma};\overline{x}_{s+1},...,\overline{x}_{s+t}) f_{\overline{\Gamma}}(\overline{\gamma}|\overline{x}_1,...,\overline{x}_s) \tag{7.27}$$

where

$$f_{\overline{\Gamma}}(\overline{\gamma}|\overline{x}_1,...,\overline{x}_s) \propto L(\overline{\gamma};\overline{x}_1,...,\overline{x}_s) f_{\overline{\Gamma}}(\overline{\gamma}) \tag{7.28}$$

It is seen from (7.27) that the information given by the posterior density function $f_{\overline{\Gamma}}(\overline{\gamma}|\overline{x}_1,...,\overline{x}_s)$, corresponding to a sample $\overline{x}_i, i = 1,...,s$ may be used as the density prior to the sample $\overline{x}_{s+1},...,\overline{x}_{s+t}$ and by this obtain the density posterior to the sample $\overline{x}_1,...,\overline{x}_s,\overline{x}_{s+1},...,\overline{x}_{s+t}$. Thus (7.27) is a straightforward tool to update information on the parameters $\overline{\Gamma}$.

The statistical model formulated above is a Bayesian statistical model based on Bayes' rule. The Bayesian method of dealing with statistical uncertainty is seen to require a choice of a prior density function for the unknown parameters $\overline{\Gamma}$. By using a prior density function which effectively may be considered as non-informative one can suppress subjectivity. The prior density function is said to be non-informative if the posterior density and the likelihood function are proportional, or almost proportional.

Beliefs or information of an objective character but not necessarily obtained in terms of a direct sample of $\overline{X}$ can be incorporated as prior information by use of a so-called natural conjugate prior, if it exists, adjoined to the density function $f_{\overline{X}|\overline{\Gamma}}(\overline{x}|\overline{\gamma})$, see e.g. Ditlevsen et al. [3].

The updated reliability may be calculated from a predictive posterior density of $\overline{X}$

$$f_{\overline{X}}(\overline{x}|\overline{x}_1,...,\overline{x}_k) = \int_{\overline{\gamma}} f_{\overline{X}|\overline{\Gamma}}(\overline{x}|\overline{\gamma}) f_{\overline{\Gamma}}(\overline{\gamma}|\overline{x}_1,...,\overline{x}_k) d\overline{\gamma} \tag{7.29}$$

which models the joint uncertainty of $\overline{X}$ arising from the inherent uncertainty and the statistical uncertainty. The idea to model estimation uncertainty in a structural reliability analysis as additional random variables and estimate a predictor reliability index based on the predictive posterior density function has been suggested in e.g. Ditlevsen [24]. It may be noticed that the applicability of the predictor reliability index has been discussed and contested in Kiureghian [26]. When the posterior density is estimated the updated reliability can be estimated using this density function.

7.3.2 Updating by Relation Information

By using FORM it is also possible to take additional information into account about variables which are not directly related to the basic variables. The procedure, repeated here, for coupling this additional information with the reliability analysis is presented in Madsen [27] and Ditlevsen et al. [3].

In a reliability analysis, the variables concerning mass and stiffness can be modelled as random variables. This means that the ith angular eigenfrequency ω_i of the structure is related to the stiffness and mass by a functional relation $\kappa(\cdot)$. A realization of the random vector $\overline{X}$ implies

$$\omega_i = \kappa(\overline{x}) \tag{7.30}$$

If e.g. vibration measurements of a structure give additional information about the eigenfrequencies this information can be incorporated into the reliability analysis by saying that $\overline{X}$ satisfies the relation

$$\Omega_i = \kappa(\overline{X}) \tag{7.31}$$

where the ith angular eigenfrequency is modelled as a random variable Ω_i which statistical characteristic is assumed to be obtained from vibration measurements.

To couple the relation information expressed by (7.31) to a reliability analysis it is convenient to introduce concepts similar to the concepts safety margin (7.3). An event function is thus defined corresponding to the definition of a failure function

$$h(\overline{x}, \omega_i) = \kappa(\overline{x}) - \omega_i = 0 \tag{7.32}$$

and an event margin is defined as

$$H(\overline{X}, \Omega_i) = \kappa(\overline{X}) - \Omega_i = 0 \tag{7.33}$$

The event margin $H(\overline{X}, \Omega_i)$ is thus of the same fundamental form as a safety margin.

The event margin in (7.33) can also be used if e.g. a crack length, a deformation etc. is measured. Further, an event margin of the type

$$H(\overline{X}, \cdot) \leq 0 \tag{7.34}$$

can be used if e.g. proof loading of the structure has been performed.

The updated probability of failure by using an event margin of the equality type in (7.33) becomes, see e.g. Madsen [27].

$$P_f = P(M \leq 0 | H_1 = ... = H_j = 0) = \frac{\frac{\partial^j P(M \leq 0 \cap H_1 - \epsilon_1 \leq 0 \cap ... \cap H_j - \epsilon_j \leq 0)}{\partial \epsilon_1 ... \partial \epsilon_j}}{\frac{\partial^j P(H_1 - \epsilon_1 \leq 0 \cap ... \cap H_j - \epsilon_j \leq 0)}{\partial \epsilon_1 ... \partial \epsilon_j}} |_{\overline{\epsilon}=0} \tag{7.35}$$

where $M \leq 0$ represents failure either at element or system level. j is the number of event margins. If the first three angular eigen frequencies are identified from the vibration measurements j is equal to three. This probability of failure (7.34) can be estimated by using the techniques available for estimating the probability of failure of a parallel system.

7.3.3 Updating Based on Vibration Measurements

In this section updating of the reliability analysis based on additional information from vibration measurements will be considered.

The updated statistical characteristics of the variables of interest are either directly connected to the basic variables, e.g. damping estimates, mass estimates and

stiffness estimates, i.e. sample information. Otherwise, the additional information can also be related to a functional relationship between the basic variables, i.e. relation information.

The framework which can be used when additional information obtained from experiments shall be coupled to prior information is the Bayesian statistical model which is presented in section 7.3.1 concerning sample information. In section 7.3.1 it is assumed that the additional information is obtained as a sample of the variables to be updated. However, when a system identification experiment is performed the additional information is not given as a sample of the parameters of interest, but as estimates of the parameters $\bar{\theta}$. This implies that the expressions in section 7.3.1 cannot be used. However, the Bayesian statistical model can be used. In chapter 2, it is explained by assuming Gaussian parameters and Gaussian measurements, that the posterior density function can be given by the updated mean value and the updated covariance matrix $\overline{\overline{C}}_{\hat{\bar{\theta}}}$ given by

$$\overline{\overline{C}}_{\hat{\bar{\theta}}} = \left(\overline{\overline{C}}^{-1}_{\hat{\bar{\theta}}_p} + \overline{\overline{\Delta}}^T \overline{\overline{C}}^{-1}_{\varepsilon\varepsilon} \overline{\overline{\Delta}} \right)^{-1} \tag{7.36}$$

where $\overline{\overline{C}}_{\hat{\bar{\theta}}_p}$ is the prior covariance matrix of the parameter $\bar{\theta}$. This expression indicates that the inverse of the updated covariance matrix can be obtained by adding the inverse of the covariance matrix of the prior information and a term which in certain circumstances corresponds to the Fisher information matrix. This corresponds to what one intuitively would have expected and it also corresponds to a result known from information theory which expresses that information is additive for independent events, see e.g. Reza [28]. This implies that the updated Fisher information matrix $\overline{\overline{J}}^u$ is given by a sum of the information matrix corresponding to prior information $\overline{\overline{J}}^0$ and an information matrix $\overline{\overline{J}}$ corresponding to new information.

$$\overline{\overline{J}}^u = \overline{\overline{J}}^0 + \overline{\overline{J}} \tag{7.37}$$

In the following (7.37) will be used assuming prior information independent of the new information.

In the following examples it is shown how the reliability can be updated based on information obtained from system identification experiments, Further, it will be seen that the expected updated structural reliability in the principle is an experiment design criterion as well as the determinant criterion etc.

7.4 Example 7.2: Optimal Choice of Sampling Interval for Identifying a SDOF System

In this example a single-degree-of-freedom (SDOF) system is considered. The aim of the example is to show how the updated reliability can be estimated from additional information obtained by performing a system identification experiment.

The system considered, shown in figure 7.6, is assumed to model a steel structure. It is assumed that the system can be modelled as a SDOF subjected to white noise.

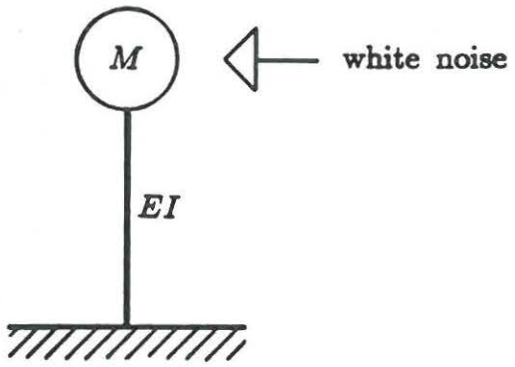

Figure 7.6: Model of a steel construction

The length, mass and stiffness are modelled such the undamped angular eigenfrequency $\omega_0 = 2\pi$ rad/sec and the damping ratio $\zeta = 0.04$.

A fatigue reliability analysis is performed to determine the most uncertain parameters and the reliability level before additional information is obtained

7.4.1 Reliability Analysis Based on Prior Information

It is assumed that the governing failure mode is the fatigue failure mode. This implies that the structure is modelled by one fatigue failure element at the clamped end of the beam model. Using the Palmgren-Miner rule in combination with SN-curves implies that the failure function is given by, see example 7.1

$$g(\overline{x},\overline{p}) = \ln(D_{Fail}) + \ln(K) - \ln(T_L) - k\ln(2\sqrt{2}) - \ln(\Gamma(1+\frac{k}{2})) \\ - \ln\frac{\sigma_s^k}{T_0} - \frac{k}{4}\ln(\frac{t}{22}) \tag{7.38}$$

Here, the expected lifetime T_L is 25 years and k is modelled as a constant, $k = 3$, and K is modelled as a random variable as $LN(6400MPa, 1024MPa)$ where LN signifies a log-normal distribution. Stress concentration is neglected. The standard deviation of the stress process and the zero-upcrossing period are estimated as described in example 7.1.

The random variables taken into account are

Variable	Designation	Distrib.	Exp. value	Var. Coeff
M	Mass	N	1.0*	0.1
EI	Stiffness	N	1.0*	0.1
ζ	Damping ratio	N	0.04	0.5
K	Constant in SN-curve	LN	1.0*	0.65

Table 7.2: Statistical characteristics (N : Normal, LN : Lognormal)

The random variables are assumed to be mutually independent. For simplicity D_{fail} is deterministically modelled and model uncertainty is neglected.

The above modelling of the structure implies that the reliability of the structure becomes $\beta = 1.09$.

Further, a sensitivity analysis has given the sensitivities shown in table 7.3

Variable	$\frac{\partial \beta}{\partial \mu_i} \frac{\mu_i}{100}$	$\frac{\partial \beta}{\partial \sigma_i} \frac{\sigma_i}{100}$
M	-0.0136	-0.00068
EI	0.0140	-0.00022
ζ	0.0182	-0.00904
K	0.0092	-0.00307

Table 7.3: Sensitivity of the element reliability index to variations of the parameters of the stochastic variables.

Table 7.3 shows that largest contributions to the overall uncertainty are due to the damping ratio and the parameter K describing the fatigue strength. In the following, the change of the updated reliability to a change of the updated variance of the damping ratio will be investigated. The updated variance of the damping ratio is calculated as function of the design variables.

7.4.2 Reliability Analysis Based on New Information

Now, it will be investigated how the reliability can be expected to change by performing a system identification experiment. It is assumed that the structure can be identified by an ARMA(2,1) model.

In chapter 5 an analytical solution for the Fisher information matrix is given as function of the sampling time Δt and the number of data N when it is assumed that an ARMA(2,1) model is used. Using this information matrix and the expression (7.37) for the updated information matrix, the updated variance of the damping ratio can be estimated. The updated variance can now be used in the reliability calculations instead of the prior variance of the damping ratio.

In figure 7.7, the expected updated reliability is show as a function of the sampling time Δt for $N = 5000$. It is assumed that the updated mean value of the damping

ratio corresponds to the prior mean value of the damping ratio. This implies that the sensitivity of the results with regard to the mean value has to be investigated. In chapter 8 such sensitivity studies will be performed.

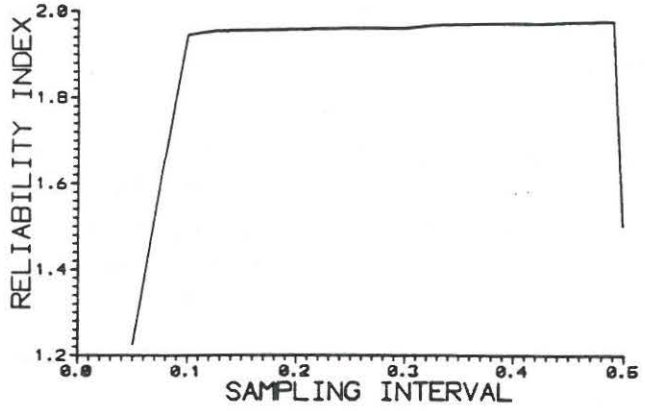

Figure 7.7: The updated reliability as function of the sampling time Δt for $N = 5000$

It is seen that the curve of the updated reliability has a maximum. This maximum corresponds to a minimum of the updated variance of the damping ratio. In chapter 5 it is shown that this minimum is obtained for $\Delta t = 0.499$.

In figure 7.8 the variation of the updated reliability index is shown as a function of the number of data points N for $\Delta t = 0.499$.

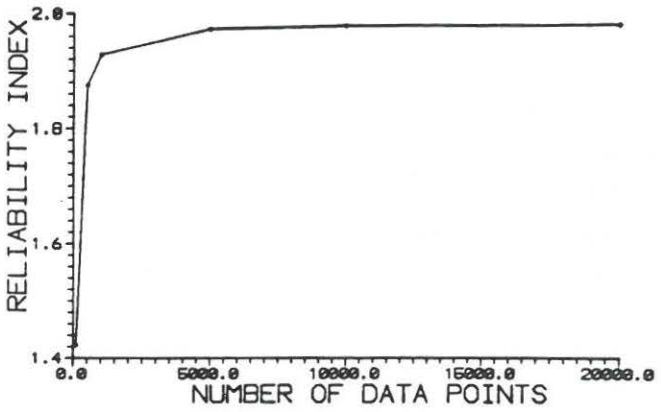

Figure 7.8: The updated reliability as function of number of data points N for $\Delta t = 0.499$

Figure 7.8 shows that only a limited improvement can be obtained when the number of data points has reached a given magnitude. In chapter 8 it will be shown how the optimal number of data points can be estimated.

It may be noticed from this example that the updated reliability can be used as a design criterion instead of e.g. the determinant criterion. It is seen that the updated reliability index is a function of the design variables. This implies that

the updated reliability could have been optimized giving an optimal sampling time.

7.5 Example 7.3: Optimal Locations of Sensors for Identifying a Beam Model

In this example the simply supported plane, vibrating Bernoulli-Euler steel beam model considered in example 6.2 is investigated. The aim of this example is the same as in the above example, i.e. to show how the updated reliability can be estimated from additional information obtained by performing a system identification experiment.

The system considered, shown in figure 7.9, is assumed to model a steel construction subjected to a white noise excitation.

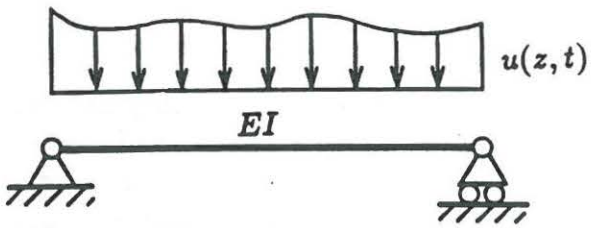

Figure 7.9: Model of a steel construction

The length, mass and stiffness are modelled as in example 6.2. This means that the undamped angular eigenfrequency of the first mode is $\omega_1 = 2\pi$ rad/sec implying that the damping ratio of the first mode is $\zeta_1 = 0.04$.

As in example 7.2 a fatigue reliability analysis is performed to determine the most uncertainty parameters and the reliability level before additional information is obtained.

7.5.1 Reliability Analysis Based on Prior Information

It is assumed that the governing failure mode is the fatigue failure mode. By using the Palmgren-Miner rule in combination with SN-curves implies that the failure function from example 7.2 can be used. The variables in the failure function and the response are modelled and calculated as described in example 6.2. Here, the response is calculated by taking the first three modes into account.

The random variables considered are

Variable	Designation	Distrib.	Exp. value	Var. Coeff
ρ_m	Mass per length	N	1.0*	0.1
EI	Stiffness	N	1.0*	0.1
ζ_1	Damping ratio	N	0.040	0.5
ζ_2	Damping ratio	N	0.010	0.8
ζ_3	Damping ratio	N	0.004	0.8
K	Constant in SN-curve	LN	1.0*	0.65

Table 7.4: Statistical characteristics (N : Normal, LN : Lognormal)

The structure is modelled as a series system with 3 fatigue failure elements where two are placed equidistantly between 0 and $L/2$ and one in $L/2$. Between the failure elements, the stochastic variables modelling K are assumed to be correlated with the correlation coefficient $\rho = 0.5$. All the others stochastic variables are separately assumed fully correlated between the failure elements.

Using the Hohenbichler approximation the system reliability index becomes $\beta^s = 1.547$.

A sensitivity analysis has given the sensitivities shown in table 7.5

Variable	$\frac{\partial \beta^s}{\partial \mu_i} \frac{\mu_i}{100}$	$\frac{\partial \beta^s}{\partial \sigma_i} \frac{\sigma_i}{100}$
ρ_m	-0.0676	-0.00007
EI	0.06810	-0.00007
ζ_1	0.05829	-0.01362
ζ_2	0.00043	-0.00029
ζ_3	0.00078	-0.00023
K	0.03383	-0.00376

Table 7.5: Sensitivity of the systems reliability to variations of the parameters of the stochastic variables

Table 7.5 shows that the largest contributions to the overall uncertainty are due to the damping ratios and the parameter K describing the fatigue strength.

7.5.2 Reliability Analysis Based on New Information

Now, an updated relability index is estimated as a function of the number of sensores N_s, the number of data points N and location of sensors. It is assumed that the damping ratios should be estimated from an experiment. The covariance of the parameter estimates is calculated by the expression (6.160). It is assumed that the number of data points $N = 1000$, and the updated variance is estimated by (7.37).

In table 7.6 the optimal location of different number of sensors is shown. The locations are determined by maximizing the determinant of the parameter covariance matrix. Further, the updated system reliabilities are shown in table 7.6.

N_s	z_1	z_2	z_3	z_4	β^s
1	0.760L				2.297
2	0.261L	0.739L			2.469
3	0.220L	0.500L	0.791L		2.566
4	0.195L	0.417L	0.583L	0.804L	2.568

Table 7.6: The updated system reliability index shown for different number of optimally located sensors.

In table 7.7 the optimal location of different number of sensors is shown again. But now the optimal location of the sensors is determined by maximizing the system reliability index.

N_s	z_1	z_2	z_3	z_4	β^s
1	0.500L				2.521
2	0.274L	0.709L			2.560
3	0.254L	0.481L	0.764L		2.568
4	0.183L	0.437L	0.643L	0.836	2.568

Table 7.7: The updated system reliability index shown for different number of optimally located sensors.

It is seen that the updated reliability is underestimated by approximately 10 % if the optimal location of one sensor is determined by the determinant criterion. However, it is seen that the same updated reliability is obatined if the number of sensors is increased. This means, that it must be more expensive to design an experiment by the determinant criterion. This implies that one of the conclusions of the examples 7.2 and 7.3 is that the expected updated structural reliability can be used as a design criterion.

7.6 Summary

Chapter 7 has been used to introduce the structural reliability theory as a tool for planning and design of experiments. The chapter has shown that:

- Modern reliability methods can be used in an uncertainty analysis to estimate a nominal element reliability level as well as a systems reliability level. It has also been shown, that the reliability methods can be used to estimate the sensitivity of the reliability to identify the most important uncertainties, by that pointing at problems for closer investigations.
- Modern reliability methods provide a rational tool for updating a reliability analysis when additonal information becomes available.
- Experiment design based on an expected updated structural reliability seems to be possible.

7.7 References

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Chapter 8

Experiment Design Based on the Expected Updated Structural Reliability

In this chapter a method for design of optimal experiments for parametric identification of structural dynamic systems is proposed. The method takes uncertainties in the experiment design problem into account in a consistent manner implying that it is possible to make decisions concerning experiment design in a rational way. This means that the experiment design is based on a probabilistic analysis instead of a traditional deterministic analysis. The experiment design problem is generally based on uncertain parameters, lack of information, predictions and information containing uncertainty. Therefore, an experiment design method based on a probabilistic analysis is interesting. Through probabilistic modelling and analysis, uncertainties may be modelled and assessed properly, and their effects on a given decision concerning the experiment design can be handled systematically. The traditional experiment design methods do not make it possible to take information containing uncertainties into account. This implies that the experiment will be designed based on incomplete information.

The proposed experiment design method is based on a preposterior analysis, well-known from the classical decision theory. A preposterior analysis can be used when the additional costs by performing an experiment have to be reflected in the design.

By using the method it is possible to consider the following problems

- Design of an optimal experiment including optimal use of time, energy and financial costs.
- Design of an optimal experiment when only a limited amount of time, energy and financial resources are available.

Further, it is also possible to consider the question

- Should additional information be obtained ? Additional information should be obtained if the additional cost for the new information is justified by an elimination of a significant part of uncertainty.

It may be noticed that the principle of the method is available for planning and design of many kinds of experiments.

In section 8.1 the method is formulated in the light of decision theory, structural reliability theory and optimization theory. It is explained that the experiment design problem solution is based on an updated structural reliability. The updated structural reliability is estimated as a function of the reduction of uncertainty in the problem. The reduction of uncertainty, expected to be obtained if the experiment is performed, is expressed as a function of the experiment design variables as described in chapter 7. Section 8.2 deals with calculation procedures. Sections 8.3 and 8.4, respectively, are concerned with simple examples showing how the method can be used for design of experiments.

8.1 Formulation of the Experiment Design Method

In the following the experiment design problem is formulated as a decision problem in the light of decision theory thoroughly presented in e.g. Ang et al. [1] and Raiffa et al. [2]. Decision analysis is the framework which can be used when decisions have to be based on uncertain information.

8.1.1 Decision Theory

The various components of a decision problem may be integrated into a formal layout as a decision tree, consisting of a sequence of decisions. In other words, the decision tree integrates the relevant components of the decision analysis in a systematic manner. The decision tree model is introduced to identify the necessary components of a decision problem consisting of:

- Feasible alternatives, including the acquisition of additional information, if appropriate.
- The possible outcomes associated with each alternative.
- The corresponding probability assignments to the outcomes.
- The consequences, measured by its utility value, associated with each combination of alternative and outcome.

In brief, the decision tree provides an organized outline of all the information used for a systematic decision analysis.

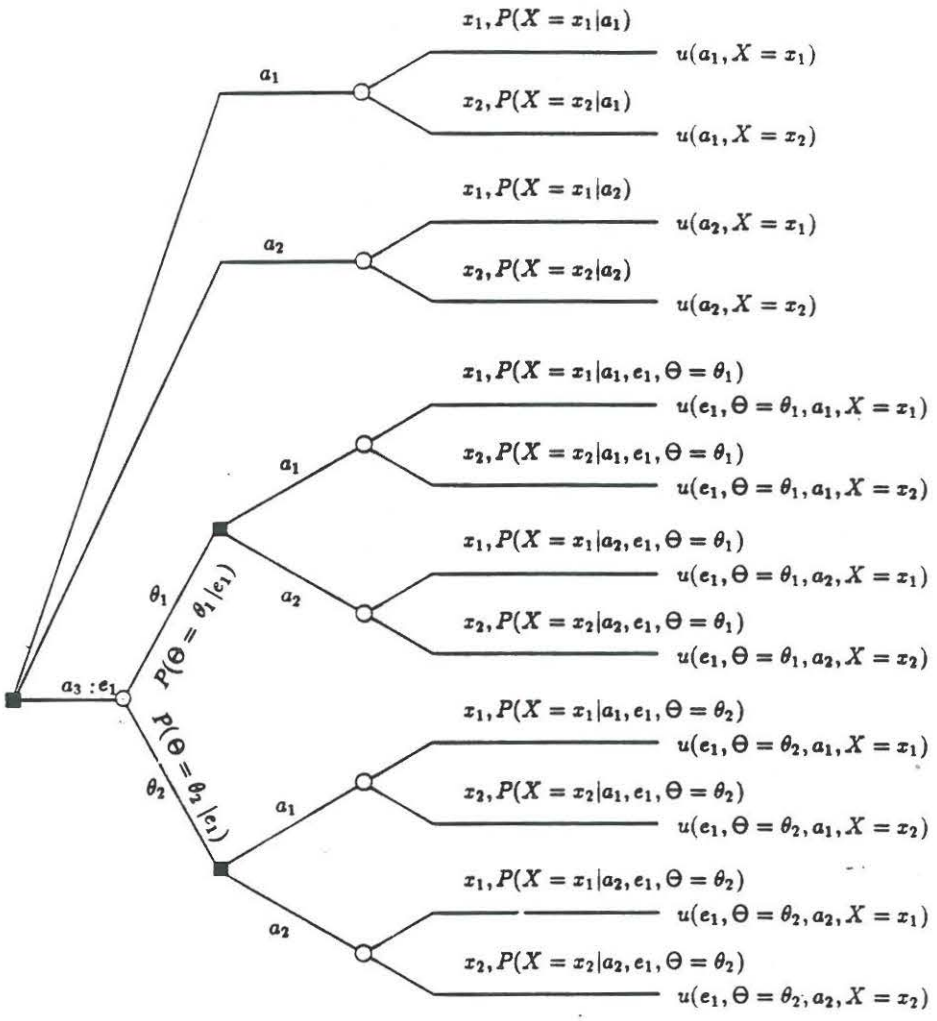

Figure 8.1 Decision tree.

The decision tree begins with a square node, called a decision node at which point there are alternatives a_1, a_2 and a_3. With each alternative, there may be several possible outcomes, shown as branches originating from a circular node, called a chance node. In figure 8.1 it is assumed that two possible outcomes x_1 and x_2 of the random variable X are associated with alternatives a_1 and a_2. The outcomes have the probability $P(X = x_j|a_i)$. The outcomes from a chance node are mutually exclusive and collectively exhaustive so that the sum of the conditional probabilities at each chance node should add up to unity. After each chance node, nature controls what will subsequently occur. Alternative a_3 is followed by experimental outcomes, whose probability will generally depend on the experiment. Hence, the probability of the experimental outcome θ_m of the random variable

Θ is conditional on the experiment e_1. The probability is $P(\Theta = \theta_m|e_1)$. Another decision node follows each experimental outcome, denoting that a decision between alternatives a_1 and a_2 will be required after observing additional information. The probability of x_j in the subsequent branches would be updated based on the particular experimental outcome. Therefore, in general this probability is expressed as $P(X = x_j|e_1, a_i, \Theta = \theta_m)$. The desirability of the consequence of each path in the decision tree is measured by its utility value recorded at the end of the sequence, such as $u(a_1, X = x_j)$ and $u(a_3, \Theta = \theta_m, X = x_j, e_1)$.

For a decision problem where an expected utility value is associated with each combination of alternative and outcome it can be shown, see Von Neumann et al. [3],

- that the alternative, among the feasible, to be selected is the alternative giving maximum of expected utility.

From figure 8.1 it is seen that the maximized expected utility $u(a^*)$ if no experiments are included into the decision model, is given by

$$u(a^*) = \max_l E_X[u(a_l, X)] \tag{8.1}$$

where a^* is the optimal alternative if the experiment is not performed. $E_X[\cdot]$ is the expectation with respect to the random variable X. $E_\Theta[\cdot]$ is the expectation with respect to the random variable Θ. If the experiment is included into the decision model the expected utility becomes

$$u(a^*, e) = E_\Theta[\max_l E''_X[u(a_l, X, e_1, \Theta)]] \tag{8.2}$$

In figure 8.1 only one experiment is assumed. If a decision tree with more than one alternative for making an experiment is considered, (8.2) becomes.

$$u(a^*, e) = \max_n E_\Theta[\max_l E''_X[u(a_l, X, e_n, \Theta)]] \tag{8.3}$$

$E''_X[\cdot]$ is the posterior expectation with respect to the random vector X.

It may be noticed that in general Θ and X are multi-dimensional vectors.

A decision analysis based entirely on existing information is called prior analysis. If such an analysis is updated subsequently with additional new information, the latter is called terminal analysis. A decision analysis with additional information is similar to the prior analysis, except that the updated probabilities, probabilities conditional on the experimental outcomes, are used in the computations. In terminal analysis, the analysis assumes that the information is available. However, additional information, obtained by performing an experiment, involves the additional time, energy and financial resources which have to be reflected in the design of the experiment. Such a decision problem calls for a so-called preposterior analysis. Design of a system identification experiment can be considered as a

decision problem involving whether and how additional information should be obtained and may then be solved by a preposterior analysis. It may be noticed that in a preposterior analysis the experimental data are not available. Instead, decisions are made based on these experimental data assumed to be obtained from the measurements, if the optimal experiment design is performed. These experimental data are estimated based on prior information and engineering judgement.

Figure 8.2 shows a decision tree for preposterior analysis. The decision tree is used in this chapter to formulate a method for optimal design of experiments for parametric identification of civil engineering structures.

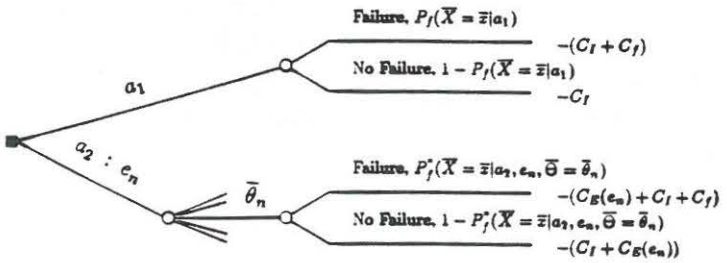

Figure 8.2: Decision tree for preposterior analysis.

The decision tree shows that at the decision node, the square node, a decision whether or not to proceed with an experiment is required, i.e. a choice between an experiment $a_2 : e_n$ and no experiment a_1. $a_2 : e_n$ indicates alternative a_2 with experiment e_n to get additional information. If an experiment is chosen the experimental outcomes of the experiment $\overline{\theta}_n$ can be obtained. $\overline{\theta}_n$ is a vector including the parameters to be determined by the experiment n. The decision tree shows that two outcomes follow the experimental outcomes, viz., structural failure and no structural failure. These alternatives are results of the outcomes of the stochastic variables included in the random vector $\overline{X}$. This means that instead of showing the outcomes, structural failure and no structural failure, the decision tree could have shown outcomes of the random vector $\overline{X}$. The posterior probability of structural failure is $P_f''(\overline{X} = \overline{x}|a_2, e_n, \overline{\Theta} = \overline{\theta}_n)$. It is also seen from the decision tree that the alternative a_1 is also followed by the two possible outcomes structural failure and no structural failure. The prior probability of structural failure is $P_f(\overline{X} = \overline{x}|a_1)$.

Here, the consequence associated with each combination of alternative and outcome is expressed by a utility value in monetary value. The utility corresponding to no structural failure and no experiment performed is the negative value of initial structural cost $-C_I$. If the cost of failure C_f is known then the expected monetary

value $C(a_1)$ corresponding to the alternative a_1 is

$$C(a_1) = -C_I - C_f P_f(\overline{X} = \overline{x}|a_1) \tag{8.4}$$

In the same way the expected monetary value $C(a_2, e)$ can be obtained for the alternative a_2 if the costs of the experiment are $C_E(e_n)$

$$C(a_2, e) = \max_n E_{\overline{\Theta}}[-C_I - C_E(e_n) - C_f P_f''(\overline{X} = \overline{x}|a_2, e_n, \overline{\Theta} = \overline{\theta}_n))] \tag{8.5}$$

where the expectation $E_{\overline{\Theta}}[\cdot]$ is obtained with respect to prior probability of $\overline{\Theta}$, i.e. before the measurements are obtained. Comparing the expected utility of a_1 (8.4) with that of a_2 (8.5), the optimal alternative at the decision node can be selected. Thus, the decision between whether an experiment should be performed or not can be made. This decision will be discussed in section 8.1.3. Before the decision at the decision node can be made the design of the experiment has to be made, in order to estimate the expected monetary value of alternative a_2. This will be discussed in the following.

8.1.2 The Experiment Design Optimization Problem

Maximum of the expected monetary value $C(a_2, e)$ is seen to be a function of the experiment e_n. The experiment is described by the experiment design variables $\overline{Z}$. In the following, the experimental design variables $\overline{Z}$ will be used instead of e_n. This means that the maximizing with respect to e_n now can be substituted with a maximizing with respect to $\overline{Z}$

An optimal experiment can be obtained by solving following optimization problem obtained from (8.5)

$$\min_{\overline{Z}} \quad C(a_2, \overline{Z}) = E_{\overline{\Theta}}[C_E(\overline{Z}) + C_f P_f''(\overline{X} = \overline{x}|a_2, \overline{Z}, \overline{\Theta} = \overline{\theta}_n)] \tag{8.6}$$

$$s.t \quad C_E(\overline{Z}) \leq C_{max}$$
$$Z_i^l \leq Z_i \leq Z_i^u \qquad i = 1, 2, ..., N_D \tag{8.7}$$

N_D is number of design variables. C_{max} is the optimal cost of the experiment. The expected monetary value $C(a_2, \overline{Z})$ is the objective function. As constraints upper and lower limits on the design variables $\overline{Z}$ and an upper limit of the costs of the experiment $C_E(\overline{Z})$ are given.

The above probabilities of structural failure $P_f(\cdot)$ and $P_f''(\cdot)$ will be estimated by using FORM presented in chapter 7. In chapter 7 it is seen that the probability of failure $P_f(\cdot)$ can be estimated by a reliability index β. This element reliability index is estimated based on a failure function $g(\overline{x})$. This means that the experiment design shall be based on a failure function corresponding to the parameters to be estimated from the measurements. This implies that a choice of a fatigue failure function is relevant here since the parameters normally estimated from full-scale

measurements on dynamically sensitive structures are related to the fatigue failure mode. Generally, this failure mode is the most important failure mode for dynamically sensitive structures which among other things can be seen from the example with the mono-tower platform in chapter 7. The fatigue failure function can be established by different damage accumulations models. E.g. the model used in the example in chapter 7 based on Miner's rule combined with the so-called S-N approach. Among others models, a crack growth model based on fracture mechanics can be mentioned where the most used law is the Paris Law, see e.g. Paris et al. [4].

The advantages associated with an experiment design based on the updated probability of failure are:

- The uncertainty of the parameters will be reduced such the total uncertainty of the performance of the system is minimized, i.e. the parameters with most influence on the performance will be estimated more accurate than parameters which have less influence on the performance, see e.g. example 7.2.
- Basing the experiment design on the updated probability of failure implies an experiment design based on an expected updated prior information.

8.1.2.1 Modelling of the Cost Function

One of the difficulties with the above optimization problem is how C_f and $C_E(\overline{Z})$ may be modelled.

When a structure fails it is necessary to pay various costs such as repair costs, reconstruction costs, clean-up costs, loss of income, costs due to loss of social prestige and possible deaths. The total cost of failure C_f may range from e.g. 2 to 5 times the initial cost of a structure, see e.g. Marshall [5]

The costs of obtaining the new information $C_E(\overline{Z})$ have to cover not only the sample records and instrumentation but also the cost of statistical analysis of the information and an appropriate share of costs of planning. A simple and useful function for the cost of an experiment is e.g. a linear function, see Ang et al. [1]. If the experiment design variables are Z_1 and Z_2 than the cost function may be modelled if a linear function is used

$$C_E(\overline{Z}) = C_0 + C_1 Z_1 + C_2 Z_2 \tag{8.8}$$

C_0 may e.g. be interpreted as representing the cost of the instrumentation and planning. C_1 may be interpreted as an additional cost per e.g. number of sample records and C_2 can e.g be the cost of an additional sensor. Sometimes a complicated cost function can be used, e.g. when a learning effect is introduced in the statistical analysis. However, a final modelling of the costs of the experiment design can first be done when the design variables $\overline{Z}$ are known.

8.1.3 Value of Information

The solution of the optimization problem in (8.6)-(8.7) implies that the maximum expected utility of the alternative a_2 can be obtained. I.e. the optimal experiment design can be provided.

The next question is; "Should the experiment be performed ?" This question can be answered by comparing the expected utility of alternatives a_1 and a_2 in the decision tree in figure 8.2.

To obtain an indicator of the expected utility by performing the optimal experiment the term value of information VI can be defined as

$$VI = C(a_2, \overline{Z}) - C_E(\overline{Z}) - C(a_1, \overline{X}) \tag{8.9}$$

which corresponds to the difference between the expected monetary value of alternative a_2, excluding the cost of the measurements, and the expected monetary value of alternative a_1. The value of information tells two things. Firstly, if VI exceeds the cost of the experiment, the experiment should be performed. Secondly by considering the value of information an indicator of the expected gain to be obtained by performing an experiment is available. It is seen from the definition of the value of information that the costs of an experiment are bounded by a limit referred to as the value of perfect information VPI. Perfect information is obtained by an experiment giving measurements from which the parameter estimates only including an inherent uncertainty can be obtained.

In the light of the formulation of the expected monetary value of the alternatives a_1 and a_2, the expected gain indicated by the value of information is an expression for the extended lifetime of the structure under consideration. But the expected gain do not include the value of the additional information produced if the information also can be used in another connection. This means that the formulation of the expression for the expected monetary for alternative a_2 depends on the purpose of the experiment. An example of an other formulation could be obtained in the following way.

Assume, that two dynamically sensitive structures based on the same concept has to be designed. Then a question of interest may be: "Should the engineer design one structure first and perform a full-scale measurement before he designs the second structure" ?. This question can be answered by going through following steps.

- The first step can e.g. be a reliability analysis to estimate the sensitivity of failure to variation of the mean values and the standard deviations of the basic variables and determinististic parameters in the structural design model. The result of an analysis gives the basic variables of interest. The results of the sensitivity analysis will tell us about the amount of a possible extension of the structures lifetime by obtaining additional information.
- Further, to investigate the amount of a possible reduction of the initial cost of the second structure we can e.g. perform a reliability based shape optimization

with the reduced number of basic variables. I.e. the basic variables of interest. The result of an analysis is the sensitivities of the expected initial cost of the structure to variation of the mean values and the standard deviations of the basic variables and determinististic parameters in the structural design model.

- The next step is design of the experiment if the sensitivities estimated from the reliability analysis and the reliability based shape optimization indicate that we can obtain an expected gain by performing an experiment on the first structure before we design the second.
- The expected monetary value of alternative a_2 is

$$C(a_2, \overline{Z}) = E_{\overline{\Theta}}[-C_I + \Delta C_I - C_E(\overline{Z}) + \Delta C_f - C_f P_f''(\overline{X} = \overline{x} | a_2, \overline{Z}, \overline{\Theta} = \overline{\theta}_n)] \tag{8.10}$$

where ΔC_I is the expected reduced initial cost of the structure number two if the optimal experiment is performed. ΔC_f is the reduced cost of failure for the second structure. The value of information based on (8.10) is an indicator expressing the amount of the expected gain that can be obtained if one structure is designed first and an optimal experiment is performed before design of the second structure.

The above discussion of the value of information shows that it is possible to get an indicator of the value of information. Based on this indicator a choice between the two alternatives a_1 and a_2 in the decision tree can be done. Of course, such a decision is conditioned on prior information. This means that the sensitivity of the choice of an optimal alternative at the decision node is of interest. Therefore, a sensitivity analysis has to be performed before final decision can be made.

8.2 Calculation Procedures

In this section it will be outlined how the proposed experiment design method can be used. It may be noticed that a preposterior analysis is used. I.e. the decisions concerning the experiment design are not based on obtained experimental data. Instead the decisions are based on the expected experimental data assumed to be obtained from the measurements, estimated based on prior information and engineering judgement.

The design method is in section 8.1 formulated as an optimization problem. It may be realized that in certain circumstances a full optimization is unrealistic. It is due to, that the optimization problem in (8.6)-(8.7) can imply expensive calculations (long calculation time) and that accurate estimates of the gradients of the objective function are generally required to achieve convergence of a mathematical optimization algorithm.

In the following it will be explained how design of an experiment can be done by a sequential procedure. This can be used if a full optimization is impossible.

Further, the sequential design procedure is presented to outline the steps in the experiment design method proposed in this chapter. Design by a sequential design procedure means that different experiment designs are chosen and then the value of information is estimated for each of these. Based on these estimates of value of information estimates the most optimal design among them can be chosen.

8.2.1 Sequential Design

The experiment design procedure can be divided into following steps

1) For the structure under consideration a structural model, prior statistical characteristics of the parameters in the model and excitation are specified based on prior information and engineering judgement.
2) Estimate the costs of structural failure.
3) The parameters to be estimated from the measurements $\overline{\Theta}$ can now be determined by different sensitivity analysis.
4) Design an experiment if the sensitivity analysis shows that it can be valuable to obtain new information. A chosen design implies that the number of sensors, the location of sensors, the excitation signal etc. are known. This means that the whole identification problem is completely specified.
5) Calculate the updated covariance matrix $\overline{\overline{C}}^{u}_{\overline{\Theta}} = E_{\overline{\Theta}}[(\overline{\overline{J}}^{u})^{-1}]$ of the parameter vector $\overline{\Theta}$ based on the updated information matrix $\overline{\overline{J}}^{u}$ (7.37). The updated covariance matrix is estimated based on the chosen model, prior information and the proposed experiment design. The expectation $E_{\overline{\Theta}}[\cdot]$ is with respect to prior statistical characteristics about the outcome of the assumed experiment. Instead of this expectation calculation the updated covariance matrix can be estimated based on a best prior mean of $\overline{\Theta}$.
6) Estimate the updated probability of failure based on the updated covariance matrix and the prior mean of the parameter vector $\overline{\Theta}$. It may be noticed that it is assumed that the updated mean value corresponds to the mean value corresponding to prior information. If the parameters $\overline{\Theta}$ are directly related to the basic variables $\overline{X}$ the updated probability of failure is directly estimated by a calculation of an updated system reliability index. On the other hand if the parameters $\overline{\Theta}$ are related to $\overline{X}$ by a functional relation the updated probability of failure can be estimated as described in the section 7.3.2 concerning updating by relation information.
7) Calculate the expected monetary value by use of (8.6).
8) Repeat 5)-7) with different proposed experiment designs.
9) Calculate the value of information by use of (8.9) for the most optimal of the proposed experiment designs and compare with the costs of the corresponding experiment. Make a sensitivity analysis of the decision with respect to the prior information.

10) Make a final decision based on the sensitivity analysis for the value of information. The final decision gives an answer to the question: "Should the most optimal experiment design among the proposed designs be performed" ?.

The steps 1)-10) outline the experiment design method proposed in this chapter. The procedure outlined can be used to choose between different proposed designs. However an optimal experiment design is obtained by solving the optimization problem in (8.6)-(8.7). This means that the steps) 4)-7) shall be repeated until convergence is achieved.

In the following two simple examples are given to show the applicability of the proposed method for optimal design of system identification experiments.

8.3 Example 8.1: Optimal Choice of Sampling Interval for Identifying a SDOF System

In this example the problem in example 7.2 is considered again. In example 7.2 it is shown how the expected updated reliability can be estimated as function of the experiment design variables Δt (sampling interval) and N (number of data points). It was found in example 7.2 that the optimal sampling time could be determined but the optimal number of data points N could not be determined. In this example it will be shown how N can be determined. This simple example is chosen since it gives a good description of the principle of the method.

The optimization problem which has to be solved becomes

$$\min_{\Delta t, N} \quad C(\Delta t, N) = E_{\overline{\Theta}}[C_E(N, N_s) + C_f P_f^{"}(\Delta t, N)] \tag{8.11}$$

$$s.t \quad \Delta t^l \leq \Delta t \leq \Delta^u$$

$$N^l \leq N \leq N^u \tag{8.12}$$

N_s is the number of sensors. Here, one sensor is assumed to be used.

From example 7.2 it is known that the optimal sampling time is $\Delta t = 0.499$. Therefore, the only optimization variable will be N.

The cost of the experiment is modelled by

$$C_E(N, N_s) = C_0 + C_1 N + C_2 N_s \tag{8.13}$$

It may be noticed that the cost is deterministically modelled.

The optimization problem (8.11)-(8.12) can in principle be solved using any general non-linear optimization algorithm. In this thesis, the optimization problems have been solved by the NLPQL algorithm developed by Schittkowski [6]. The NLPQL algorithm is based on the optimization method by Han, Powell and Wilson, see Gill et al. [7]. The algorithm is an effective method where each iteration consists

of two steps. The first step is a determination of the search direction. The second step is a line search.

The NLPQL algorithm requires estimates of the gradients of the objective function $C(N)$. The estimation of the gradients of the objective function can be made as described in the following.

The derivative of the objective function $C(N)$ with respect to the design variable N is

$$\frac{\partial C(N)}{\partial N} = \frac{\partial P_f''(N)}{\partial N} C_f + \frac{\partial C_E(N, N_s)}{\partial N} \tag{8.14}$$

The last term in (8.14) is easy to estimate analytically but the gradients of the updated probability of failure are more difficult to obtain.

Since the probability of failure is estimated by FORM the gradient of the objective function becomes

$$\frac{\partial C(N)}{\partial N} \approx \varphi(-\beta) \frac{\partial(-\beta)}{\partial N} C_f + C_1 \tag{8.15}$$

where $\varphi(\cdot)$ is the standard normal density function.

The derivative of the reliability index with respect to the design variable is

$$\frac{\partial \beta}{\partial N} = \sum_{j=1}^{n_{\overline{\Theta}}} \frac{\partial \beta}{\partial \sigma_{\Theta_j}} \frac{\partial \sigma_{\Theta_j}}{\partial N} \tag{8.16}$$

where σ_{Θ_j} is the standard deviation of the jth parameter Θ_j in the parameter vector $\overline{\Theta}$ containing $n_{\overline{\theta}}$ parameters. Here, $n_{\overline{\theta}} = 1$ since the only parameter to be determined from the experiment is the damping ratio. $\frac{\partial \beta}{\partial \sigma_{\Theta_j}}$ is determined by (7.18). The derivative $\frac{\sigma_{\Theta_j}}{\partial N}$ can be estimated numerically from the expression for the updated information matrix $\overline{\overline{J}}^u$. The inverse of the updated information matrix includes the standard deviation σ_{Θ_j} of the jth parameter Θ_j. The updated information matrix is determined by the analytical expression from chapter 5.

8.3.1 Results

The optimal solution of N^{opt} for an experiment with one sensor for identifying the SDOF can now be estimated.

It is assumed that the cost function can be modelled as:

$$C_0 = 10^4 \ DKK., \quad C_1 = 5 \ DKK., \quad C_2 = 10^4 \ DKK. \tag{8.17}$$

where DKK is Danish kroner. The optimal solution of N for various values of cost of failure C_f is shown in table 8.1.

C_f (DKK.)	$P_f''(\cdot)$	$C(\cdot)$ (DKK.)	$C_E(\cdot)$ (DKK.)	N^{opt}
10^5	$3.06\,10^{-2}$	$2.40\,10^4$	$2.09\,10^4$	181
10^6	$2.59\,10^{-2}$	$4.86\,10^4$	$2.27\,10^4$	534
10^7	$2.44\,10^{-2}$	$2.72\,10^5$	$2.80\,10^4$	1600
10^8	$2.38\,10^{-2}$	$2.43\,10^6$	$4.50\,10^4$	5010
10^9	$2.36\,10^{-2}$	$2.38\,10^7$	$1.02\,10^5$	16500

Table 8.1: The optimal solution of N for various values of cost of failure C_f.

It is seen, as expected, that N^{opt} increases when C_f increases, which means that acquisition of more information is of course more relevant when cost of failure increases.

In table 8.2 the value of information corresponding to the values in table 8.1 is shown. This value of information VI represents the maximum cost that may be allowed for acquisition of additional information. If VI exceeds the cost of experiment C_E, the experiment should be performed. The probability of failure $P_f(\cdot)$ corresponding to the prior information is estimated in chapter 7.

$C_f(DKK.)$	$C_f P_f''(\cdot)(DKK.)$	$C_f P_f(\cdot)(DKK.)$	$C_E(\cdot)(DKK.)$	$VI(DKK.)$
10^5	$3.06\,10^3$	$1.38\,10^4$	$2.09\,10^4$	$1.07\,10^4$
10^6	$2.60\,10^4$	$1.38\,10^5$	$2.27\,10^4$	$1.12\,10^5$
10^7	$2.44\,10^5$	$1.38\,10^6$	$2.80\,10^4$	$1.14\,10^6$
10^8	$2.38\,10^6$	$1.38\,10^7$	$4.50\,10^4$	$1.14\,10^7$
10^9	$2.37\,10^7$	$1.38\,10^8$	$1.02\,10^5$	$1.14\,10^8$

Table 8.2: Value of information.

It is seen that the experiment should be performed if the cost of failure of the structure is larger than $10^6\ DKK$.

However, before a final decision is reached further investigations must be performed. It may be noticed that the results obtained above are estimated based on prior information and information which is assumed to be obtained by performing the experiment. This means that the expected monetary value $C(N)$ is an uncertain quantity.

To investigate the sensitivity of $C(N)$ with respect to variations of the information, which is used above, a sensitivity study is made.

Table 8.3 shows the sensitivities of the expected monetary value $C(N)$ to variation of the mean value and standard deviation for the variables which are stochastically modelled. The sensitivity of $C(N)$ with respect to a mean value μ_j is

$$\frac{\partial C(N)}{\partial \mu_j} \approx \varphi(-\beta)\frac{\partial(-\beta)}{\partial \mu_j}C_f + \frac{\partial C_E(N, N_s)}{\partial \mu_j} \tag{8.18}$$

The derivative with respect to the standard deviation can be obtained in the same manner. It is seen from (8.18) that the sensitivities of $C(N)$ is proportional to the sensitivities of the reliability index. The sensitivities of the reliability index are given in table 8.3 for an optimal value of N. Here, the sensitivities are shown for the situation where it is assumed that cost of failure $C_f = 10^7$ DKK, i.e $N^{opt} = 1600$. It may be noticed that the sensitivities shown in table 8.3 and the sensitivities shown in table 7.2 are different. This is caused of that the standard deviations of the damping ratios are different. The standard deviation used in example 7.2 corresponds to the prior information while the standard deviation used here corresponds to the updated standard deviation.

Variable	$\frac{\partial \beta}{\partial \mu_j}\frac{\mu_j}{100}$	$\frac{\partial \beta}{\partial \sigma_j}\frac{\sigma_j}{100}$
M	-0.0311	-0.00156
EI	0.0356	-0.00248
ζ	0.0224	-0.00023
K	0.0317	-0.01699

Table 8.3: The sensitivities of $C(N)$ to variations of the mean values μ_j and standard deviations σ_j for the basic variables.

Table 8.3 shows that the dominating contributions to the overall uncertainty are due to the parameter K describing the fatigue strength. Further, if the results in table 8.3 are compared with the results in table 7.2, it is seen that the contribution of the damping ratio to the overall uncertainty is reduced while the contributions of the mass and the stiffness are increased. However, K has the largest contribution. This means that K is a variable with a large influence on the optimal design of the experiment and the value of information. Therefore, the influence of the statistical characteristic of K on the experiment design will be investigated closer.

In figure 8.3 the value of information minus the cost of the experiment $(VI - C_E(\cdot))$ is shown to variations of the mean value μ_K of K and to variations of the coefficient of variation δ_K.

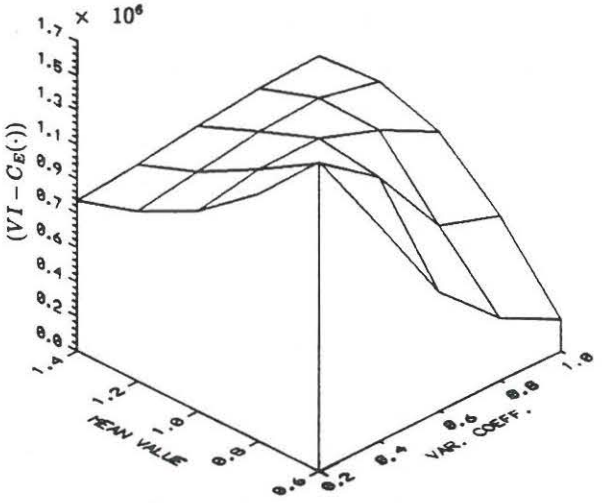

Figure 8.3: The sensitivity of $(VI - C_E(\cdot))$ to variations of the mean value μ_K of K and to variations of the coefficient of variation δ_K, $C_0 = 10^4\ DKK$.

Figure 8.3 shows that the value of information minus the cost of the experiment $(VI - C_E(\cdot))$ is positive for all values of the mean value and coefficient of variation. This means that the experiment should be performed. However, it is also seen that $(VI - C_E(\cdot))$ is very sensitive to variations of the mean value and the coefficient of variation. Especially, when $\mu_K < 1.0$ and $\delta_K > 0.4$. This implies that one has to take more care when a decision concerning performing an experiment is based on values in this area.

In figure 8.4 it is assumed that an experiment with a cost of instrumentation and planning at $C_0 = 10^6\ DKK$ is used.

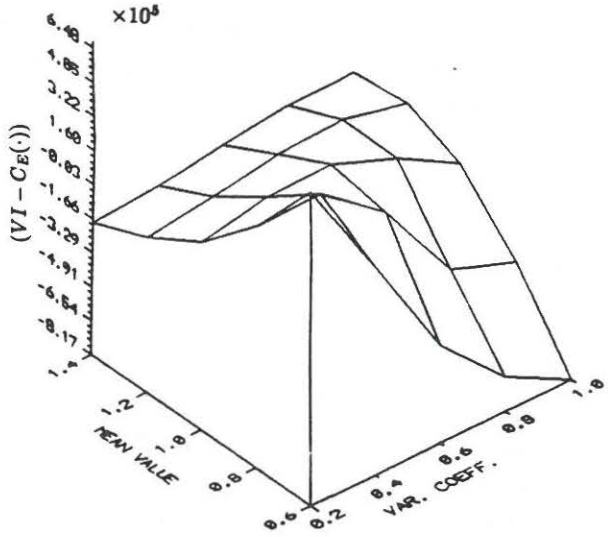

Figure 8.4: The sensitivity of $(VI - C_E(\cdot))$ to variations of the mean value μ_K of K and to variations of the coefficient of variation δ_K, $C_0 = 10^6\ DKK$.

It is seen from figure 8.4, that the experiment should not be performed if the mean value μ_K is small and the coefficient of variation δ_K is large.

In figure 8.5 the sensitivity of the optimal design, N^{opt} is shown to variations of the mean value μ_K and the coefficient of variation δ_K. It is seen that the optimal design is sensitive to the mean value μ_K and the coefficient of variation δ_K. Especially, for a coefficient of variation less than 0.4.

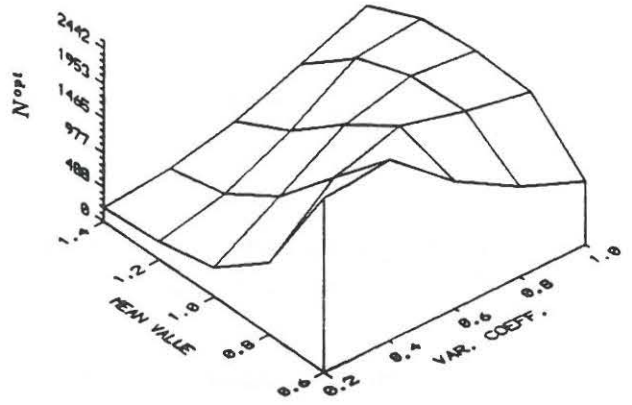

Figure 8.5: The sensitivity of the optimal design, N^{opt}.

The conclusion of the example is that the experiment should be performed if the cost of structural failure is larger than $10^6 DKK$. However, if the cost of the experiment increases, the mean value and the coefficient of variation should have more attention before a final decision should be made. It may also be noticed that an experiment giving additional information about the random variable K could be performed. It could be interesting to investigate whether additional information should be obtained by performing a system identification experiment, or an experiment, giving additional information about K. Such an investigation could also be formulated as a decision problem.

8.4 Example 8.2: Optimal Locations of Sensors for Identifying a Beam Model

In example 7.3 the expected updated reliability index is estimated for different number of sensors. It is seen that the increase in the updated reliability index is small for $N_s > 2$. One can ask: "How many sensors should be used" ?. The increased cost for an additional sensor may be justified if it eliminates a significant part of the uncertainty, thus leading to a larger expected structural reliability. Therefore, the increased cost of an additional sensor should be reflected in an experiment design method as proposed in this chapter.

In this example the problem in example 7.3 is considered. It will be shown how the optimal number of sensors can be determined. The optimization problem to solve is

$$\min_{N_s,N,\overline{z}} \quad C(N_s,N,\overline{z}) = E_{\overline{\Theta}}[C_E(N,N_s) + C_f P_f^{"}(N_s,N,\overline{z})] \tag{8.19}$$

$$s.t \quad z_i^l \leq z_i \leq z_i^u$$
$$N_s^l \leq N_s \leq N_s^u$$
$$N^l \leq N \leq N^u \tag{8.20}$$

$\overline{z}$ is a vector including the location of the sensors. $P_f^{"}(\cdot)$ is the updated probability of failure calculated as described in chapter 7.

The costs of the experiment are again modelled by

$$C_E(N,N_s) = C_0 + C_1 N + C_2 N_s \tag{8.21}$$

8.4.1 Results

This optimization problem (8.19)-(8.20) is solved for different values of cost of failure C_f in order to determine the optimal number of sensors. It may be noticed that the derivative of the objective function (8.19) with respect to the design variables has to be determined. This can be done as described in example 8.1. by (8.14), (8.15) and (8.16). However, in example 8.1 an element reliability index is considered. In this example a system reliability index is considered. This means that the derivative with respect to a distribution parameter in (8.16) is estimated by (7.15).

It is assumed that the cost function can be modelled as:

$$C_0 = 10^6 \ DKK., \quad C_1 = 5 \ DKK., \quad C_2 = 10^5 \ DKK. \tag{8.22}$$

In table 8.4 the expected monetary value $C(N_s,N,\overline{z})$ corresponding to optimal solutions, is shown for different values of cost of failure and number of the sensors.

C_f	$N_s = 1$	$N_s = 2$	$N_s = 3$	$N_s = 4$
10^5	$1.105\,10^6$	$1.215\,10^6$	$1.310\,10^6$	$1.410\,10^6$
10^6	$1.110\,10^6$	$1.210\,10^6$	$1.310\,10^6$	$1.410\,10^6$
10^7	$1.163\,10^6$	$1.257\,10^6$	$1.355\,10^6$	$1.456\,10^6$
10^8	$1.692\,10^6$	$1.728\,10^6$	$1.814\,10^6$	$1.914\,10^6$
10^9	$6.970\,10^6$	$6.436\,10^6$	$6.387\,10^6$	$6.487\,10^6$

Table 8.4: The expected monetary value $C(N_s,N,\overline{z})$ corresponding to optimal solutions.

Table 8.4 indicates that the optimal number of sensors is one, except for the situation, where the cost of failure $C_f = 10^9 DKK$.

In table 8.4 it is assumed that the cost of an additional sensor is $10^5 DKK$. In table 8.5 the same results are shown if the cost of an additional sensor is $10^4 DKK$.

C_f	$N_s = 1$	$N_s = 2$	$N_s = 3$	$N_s = 4$
10^5	$1.005\,10^6$	$1.106\,10^6$	$1.156\,10^6$	$1.206\,10^6$
10^6	$1.068\,10^6$	$1.110\,10^6$	$1.160\,10^6$	$1.210\,10^6$
10^7	$1.114\,10^6$	$1.157\,10^6$	$1.206\,10^6$	$1.256\,10^6$
10^8	$1.642\,10^6$	$1.628\,10^6$	$1.664\,10^6$	$1.714\,10^6$
10^9	$6.920\,10^6$	$6.336\,10^6$	$6.237\,10^6$	$6.287\,10^6$

Table 8.5: The expected monetary value $C(N_s, N, \overline{z})$ corresponding to optimal solution.

Table 8.5 shows that the optimal number of sensors is increasing when the cost of an additional sensor is decreasing. It is seen that the optimal number of sensors is one if the cost of failure $C_f < 10^7$. If $C_f = 10^8$ two sensors become optimal. For $C_f = 10^9$ three sensors are optimal.

It may be noticed that the optimal number of data points also have been determined by using the optimization problem in (8.19)-(8.20) but they are not shown here.

At last in this example the sensitivity of the expected monetary value to variations of the parameters describing the stochastic variables will be considered. In table 8.6 the sensitivities are shown for the optimal design corresponding to $C_f = 10^8$.

Variable	$\frac{\partial \beta}{\partial \mu_i} \frac{\mu_i}{100}$	$\frac{\partial \beta}{\partial \sigma_i} \frac{\sigma_i}{100}$
ρ_m	-0.0314	-0.00006
EI	0.0315	-0.00007
ζ_1	0.0274	-0.00729
ζ_2	0.0012	-0.00042
ζ_3	0.0001	-0.00002
K	0.0170	-0.00312

Table 8.6: The sensitivities of $C(N)$ to variations of the mean values μ_j and standard deviations σ_j for the basic variables.

Table 8.6 shows that the dominating contribution to the overall uncertainty is due to the damping ratio ζ_1 of the first mode and the strength parameter K. Again, this mean, that these quantities have a large influence on the optimal design and therefore should be investigated closer before a final design is chosen.

8.5 Summary

Chapter 8 has been devoted to a presentation of an experiment design method based on a preposterior analysis. The applicability of the method has been presented, see also Kirkegaard et al. [8] and Kirkegaard [9]. The characteristics features of the method are as follows:

- The method is based on a preposterior analysis implying that uncertain quantities can be modelled as random variables.
- In a preposterior analysis it is possible to consider the problem evolving whether and how additional information should be obtained.
- The method provides experiment designs reflecting the costs of the experiment and the value of additional information.
- The experiment design is based on an expected updated a priori information.

8.6 References

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Chapter 9

Conclusions

This thesis has attempted to outline how design of experiments for parametric identification of civil engineering structures can be performed. The most important subjects and methods have been discussed.

This chapter contains a summary of the thesis and general conclusions. In section 9.1, a general summary of the individual chapters is given and, in section 9.2, a general conclusion of the thesis is given. Finally, suggestions for further research are given.

9.1 Summary of the Thesis

In chapter 1, the general problem concerning design of system identification experiments has been presented. It has been explained that it is a topic mainly developed in electrical engineering. The most fundamental principles of system identification experiment design methods have been presented. Traditionally, an optimal experiment design is obtained by minimizing a scalar measure of an estimated parameter covariance matrix. This matrix is estimated based on prior knowledge, engineering judgement and choice of the experiment design variables. The aim of experiment design is to ensure that the design variables are chosen so the experiment is maximally informative about some intended application. Normally, an experiment design is chosen to be maximally informative in a Fisherian sense, i.e. the estimate of the parameter covariance matrix is assumed to go against the Cramer-Rao lower bound. This assumes, that an unbiased efficient estimator is used.

System identification experiment design has only been given little attention from structural engineers though it is an important topic for engineers working with system identification. It is an important topic, because the achievable accuracy of parameters in parametric identification is related to the experimental conditions.

In chapter 1 the scope of the work in this project is formulated as:

- Presentation of techniques available for design of experiments for parametric identification of civil engineering structures.
- Development of an experiment design method reflecting the costs of the experiment and the value of information, expected to be obtained.

Chapter 2 gives to an outline of system identification. Different models are presented and parametric estimation is defined. However, the main purpose of the chapter is to explain how estimates of the parameter covariance matrix can be obtained. The well-known Fisher information matrix, the inverse of the parameter covariance matrix for an unbiased and efficient estimator is presented. Further, an asymptotic parameter covariance matrix can be estimated if the measurements are independent random variables with zero mean. For additive Gaussian measurement noise, the inverse of the Fisher matrix and the asymptotic matrix will coincide. It may be noticed that these estimates of the parameter covariance matrix can be obtained by a single measured time series. The applicability of the asymptotic parameter covariance matrix is investigated in an example. The asymptotic covariance matrix seems to apply also for a finite number of measurements.

Chapter 3 is used to explain how system identification can be applied to parametric identification of civil engineering structures. Generally, used models and excitation types are presented. A linear, time-invariant lumped mass model seems the most frequently used model. The most frequently used type of excitation is the ambient excitation implying that a white noise assumption is used. Frequency and time domain system identification techniques are presented. The conclusion of the chapter is that time domain techniques based on e.g. ARMA models are commonly used, since they in many situations are superior to frequency domain techniques.

In the chapters 4, 5 and 6 different experiment design methods are presented and investigated.

In Chapter 4 the optimal choice of input signals is considered. Two different aspects are associated with the choice of input signal. One concerns the properties of the input signal such as its spectrum. The other concerns the shape of the input signal. The minimal requirement for an input signal is that the input signal has to be sufficiently rich to excite all modes of interest during the experiment. Design of input signal is shown to be related to design of input signal for control of dynamic systems. An optimal input signal for parametric identification of a dynamic system is traditionally based on a scalar measure of the inverse of the Fisher information matrix. Design of input signal can be made by using a time-domain approach or a frequency approach. However, design is more easily handled in the frequency domain than in the time domain when stationary solutions are usable. In an example, the optimal frequency of a stationary input signal is estimated for identification of a single-degree-of-freedom dynamic system.

In chapter 5 the choice of optimal sampling interval and experiment length is considered. Problems, such as aliasing, which can occur by a discretisation of a continuous signal are discussed. Three examples are given. The first example is

concerned with the joint optimal determination of the input spectrum, presampling filter and sampling interval. In the second example a single-degree-of-freedom system is assumed to be identified by an ARMA model. It is shown that an optimal sampling interval exists if the total number of sample points is fixed. It is far worse to use a too large sampling interval than a too small one, since the losses of information increase rapidly when the sampling interval increases from the optimal value. Further, it is shown that an optimal number of data points cannot be determined. However, it is shown that when the experiment length has reached a given magnitude only limited improvement can be obtained by increasing the number of data points. The last example, in chapter 5, is given to explain how the influence of the experiment length on the uncertainty of autospectral estimates can be considered.

Chapter 6 deals with different methods proposed for determining optimal locations of sensors when parametric identification of dynamic systems is considered. The methods are described and used in examples to investigate their applicability. Nearly all the methods are based on a scalar of an estimated covariance matrix. Further, a method based on the concept of entropy is considered. Examples are given to show the applicability of the methods. In one example it is found that design of experiments based on purely heuristic grounds may be difficult, since the optimal location of sensors can depend in a complex manner on the actual parameter values and the excitation. Another example shows that experiment design concerning structural systems can hardly be based on design methods assuming statistically independent measurements. However, nearly all the methods available for experiment design assume statistically independent measurements. When the measurements can be modelled as statistically independent random variables it is much simpler to estimate e.g. the Fisher information matrix. This is impossible if the measurements are modelled as dependent random variables. In an example, the concept of entropy is shown to measure something which is not connected with the aim of experiment design for parametric identification. At the end of chapter 6, it is proposed how to obtain an estimate of the parameter covariance matrix if the measurements are assumed to be correlated in the space parameter and independent in the time parameter.

In chapter 7, the structural reliability theory is presented. The reliability theory is mainly developed for estimating reliability of structural systems. However, it is also an excellent tool for determination of important sources of uncertainty. The applicability of the reliability methods is shown in an example. Further, the reliability methods also provide a rational tool for updating a reliability analysis. In chapter 7 it is shown how a reliability analysis can be updated when additional information from a system identification experiment becomes available. In two examples it is shown how the updated reliability of a structure can be estimated as a function of the experimental design variables. This implies that the updated structural reliability, which is expected to be obtained if the experiment is performed, is introduced as an experiment design criterion. Chapter 7 is mainly written to explain how the updated reliability of structures can be estimated as a function of the design variables. In chapter 8, this connection is used in a so-called

preposterior analysis.

In chapter 8 an experiment design method is proposed. The method is based on a preposterior analysis, well-known from the classical decision theory. Design of experiments calls for a preposterior analysis when the time, energy and financial resources added encumbered with additional information, have to be reflected in the experiment design. Since full-scale measuring on a civil engineering structure can be expensive, it is important that the costs can be reflected in the experiment design. A preposterior analysis implies that the uncertain quantities of the experiment design problem can be modelled as random variables. This means that the experiment design will be based on a probabilistic analysis. The method implies two things. Firstly, it is possible to design experiments based on complete prior information, since the uncertain quantities are modelled as random variables. Secondly, experiments using unnecessary resources can be avoided. Further, it becomes possible to answer the question: Should additional information be obtained ? In chapter 8 two examples are given to show the applicability of the proposed experiment design method.

9.2 General Conclusions and Comments

In this thesis, the topic design of experiments for parametric identification of civil engineering structures is presented. It is found that the topic has only been given little attention from structural engineers working with system identification. This attention seems to increase, due to the fact, that careful experiment design yielding data with good information is the basis of a successful identification application. Examples in this thesis show that it often suffices to base a design on intuitive reasoning. On the other hand, it is also shown that e.g. the optimal location of sensors can appear to depend in a more complex manner of the actual parameter values of the system and the excitation. However, the main thing to realize, in practice, is that the experimental conditions are directly related to the achievable accuracy in the parametric identification.

If an optimal experiment design for parametric identification is desirable then it is shown that appropriate experiment design methods exist. Such methods are based on an estimate of the parameter covariance matrix giving standard deviations expected to be obtained if the experiment is performed. However, few of these methods are devoted to experiments where the measurements can be modelled as a random field with non-trivial covariance functions. Nearly all papers concerning experiment design are based on measurements modelled as statistically independent random variables. However, the response of real engineering structures can hardly be modelled as statistically independent random variables. One of the contributions of this thesis has been examples, concerning the optimal location of sensors, where the measurements have not been modelled as statistically independent random variables. In one of the examples, it is proposed how estimates of the parameter covariance matrix can be obtained if the measurements

are assumed to be modelled as dependent quantities. The main contribution of this thesis is an experiment design method, based on a preposterior analysis, taking the costs of the experiment into account. Since full-scale measurements of civil engineering structures can be expensive, it is of great value that the experiment design reflects the costs of the experiment and the value of additional information. Further, the development of this method in the future seems an interesting topic of research. However, the broader use of experiment design methods in civil engineering require further research within following topics:

- Planning and design of experiments for parametric identification of civil engineering structures based on a preposterior analysis.
- Estimation of the parameter covariance matrix when the measurements cannot be modelled as statistically independent random variables.
- Application of the experiment design methods in real examples.
- An investigation of the identifiability problem in relation to experiment design.